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ON A HIDDEN PARAMETER IN FRIEDMAN EQUATIONS

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Abstract. Authors of this paper recently applied the theory of regularly varying functions in asymptotic analysis of cosmological parameters for the expanding universe. According to this theory, all cosmological parameters depend on a a 0-function $\varepsilon(t)$ which is "hidden" in the integral representation of regularly varying functions. We derived a differential equation for $\varepsilon(t)$ and discussed possible solutions.

1. INTRODUCTION

In our paper [Mijajlović et al., 2012], we applied the theory of regularly varying functions in asymptotic analysis of cosmological parameters for the expanding universe. For this analysis we used Friedman equations [see Friedman, 1924], which are derived from the Einstein field equations:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}, \qquad \text{Friedman equation,}$$
$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right), \qquad \text{Acceleration equation,}$$
$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0, \qquad \text{Fluid equation.}$$

Cosmological parameters appearing in these equations are:

$a = a(t)$, the scale factor,
$\rho = \rho(t)$, the energy density, and
$p = p(t)$, the pressure of the material in the universe.

It appears that these parameters, including Hubble parameter, $H(t)$, and deceleration parameter, $q(t)$, are regularly varying functions. According to the representation theory for regularly varying functions, all these parameters depend on a 0-function $\varepsilon(t)$ which is "hidden" in the integral representation of regularly varying functions. Parameters $a(t)$, $\rho(t)$ and $H(t)$ uniformly depend only on $\varepsilon(t)$, but the parameters $p(t)$ and $q(t)$ depend on $\dot{\varepsilon}$ as well. While $\varepsilon(t) \to 0$ as $t \to \infty$, this is not necessary for

$\dot{\varepsilon}$, what may lead to various evolutions of these parameters. We derive a differential equation for $\varepsilon(t)$ and discuss possible solutions.

2. REGULAR VARIATION

We shall review briefly basic notions related to regular variation. We shall need particularly properties of regularly varying solutions of the following second order differential equation

$$\ddot{y} + f(t)y = 0, \quad f(t) \text{ is continuous on } [\alpha, \infty]. \tag{1}$$

Observe that the acceleration equation has the form (1). In short, the notion of a regular variation is a form of the power law distribution, described by the following relationship between quantities F and t:

$$F(t) = t^r(\alpha + o(1)), \quad \alpha, r \in \mathbb{R}. \tag{2}$$

Obviously, the most simple form of the power law is given by the equation $y = t^k$. It is said that two quantities y and t^r satisfy the power law if they are related by a proportion, i.e. there is a constant α so that $y = \alpha t^r$. This definition of power law can be naturally extended by use of the notion of slowly varying function introduced by J. Karamata, (1930).

A real positive continuous function $L(t)$ defined for $x > x_0$ which satisfies

$$\frac{L(\lambda t)}{L(t)} \to 1 \quad \text{as} \quad t \to \infty, \quad \text{for each real } \lambda > 0. \tag{3}$$

is called a slowly varying function. A physical quantity F(t) is said to satisfy the generalized power law if

$$F(t) = t^r L(t) \tag{4}$$

where $L(t)$ is a slowly varying function and r is a real constant. So to say that $F(t)$ is regularly varying is the same as $F(t)$ to satisfy the generalized power law. The simplest example of slowly varying functions are $\ln(x)$ and iterated logarithmic functions $\ln(\ldots\ln(x)\ldots)$.

Regularly varying function have the following representation. Namely, a function L is slowly varying if and only if there are measurable functions $h(x)$ and a zero function ε and $b \in \mathbb{R}$ so that

$$L(x) = h(x)e^{\int_b^x \frac{\varepsilon(t)}{t}dt}, \quad x \geq b, \tag{5}$$

and $h(x) \to h_0$ as $x \to \infty$, h_0 is a positive constant. For further properties of regularly varying functions, one may consult Bingham et al., 1987.

As we are dealing with solutions L of differential equations which model mechanical phenomena, it is quite safe to assume that L is a twice differentiable function. The function $\varepsilon(t)$ is not uniquely determined and in our context we shall call it a hidden parameter. We shall also assume that $L(t)$ is normalized, i.e. that $h(x)$ is a constant function. The class of normalized slowly varying functions will be denoted by $\mathcal{N}$. We shall see that the fundamental cosmological parameters depend essentially on the hidden parameter $\varepsilon(t)$.

For our study of Friedman equations we need several results on solutions of equation (1). There are various conditions for $f(t)$ that ensure that regularly varying solutions of $\ddot{y} + f(t)y = 0$ exist. We shall particularly use the following result, due to Howard and Marić, see Marić, 2000 and Kusano-Marić, 2010:

Theorem *Let* $-\infty < \Gamma < 1/4$, *and let* $\alpha_1 < \alpha_2$ *be two roots of the equation*

$$x^2 - x + \Gamma = 0. \tag{6}$$

Further let L_i, *i=1,2 denote two normalized slowly varying functions. Then there are two linearly independent regularly varying solutions of* $\ddot{y} + f(t)y = 0$ *of the form*

$$y_i(t) = t^{\alpha_i} L_i(t), \quad i = 1, 2, \tag{7}$$

if and only if $\mathbf{M}(f) = \lim_{x\to\infty} x \int_x^\infty f(t)dt = \Gamma$. *Moreover,* $L_2(t) \sim \dfrac{1}{(1-2\alpha_1)L_1(t)}$.

The limit of the integral in the theorem is crucial in our analysis and it is not always easy to compute.

As $\lim_{t\to\infty} t^2 f(t) = \Gamma$ implies $\lim_{x\to\infty} x \int_x^\infty f(t)dt = \Gamma$, we see that

$$\lim_{t\to\infty} t^2 f(t) = \Gamma \tag{8}$$

gives a useful sufficient condition for the existence of regularly varying solutions of the equation $\ddot{y} + f(t)y = 0$ as described in the previous theorem. By r.v. we shall denote the term "regularly varying".

3. R. V. SOLUTIONS OF FRIEDMAN EQUATIONS

As noted, the acceleration equation obviously has the form (1) so under appropriate assumptions, i.e. that the functions we encounter are continuously differentiable as many times as necessary, the analysis of the previous section, in particular the Howard-Marić theorem, can be applied to it. For this reason, we shall write from now on the acceleration equation in the form

$$\ddot{a} + \frac{\mu(t)}{t^2} a = 0, \tag{9}$$

where

$$\mu(t) = \frac{4\pi G}{3} t^2 \left(\rho + \frac{3p}{c^2}\right). \tag{10}$$

We found in [Mijajlovic et al. 2012] r.v. solutions of Friedman equations and determined cosmological parameters:

Scale factor $a(t)$: $a(t) = t^\alpha L(t)$, where $\alpha \neq 0$ with L having the ε-representation as in Karamata Representation theorem.

Hubble parameter $H(t)$:

$$H(t) = \frac{\alpha}{t} + \frac{\varepsilon}{t}. \tag{11}$$

Deceleration parameter $q(t)$:

$$q(t) = \frac{\mu(t)}{\alpha^2}(1+\eta) = \frac{1-\alpha}{\alpha} - \frac{t\dot{\varepsilon}}{\alpha^2}(1+\eta) + \tau, \tag{12}$$

where η and τ are zero functions.

Now we introduce a new constant w related to the scale factor $a(t)$ which satisfy the generalized power law. It appears that w is in fact the equation of state parameter. So assume $a(t) = t^\alpha L(t)$, $L \in \mathcal{N}$ and $\alpha \neq 0$. We define w by

$$w \equiv w_\alpha = \frac{2}{3\alpha} - 1 \quad \text{(equation of state parameter).} \tag{13}$$

Then cosmological parameters can be put in a more standard form, widely found in the cosmology literature.

$$\begin{aligned} \alpha &= \frac{2}{3(1+w)}, & a(t) &= a_0 L(t) t^{\frac{2}{3(1+w)}} \\ H(t) &\sim \frac{2}{3(1+w)t}, & \mathbf{M}(q) &= \frac{1+3w}{2} \end{aligned} \tag{14}$$

By the representation theorem for regularly varying functions we see that the scale factor $a(t)$ and the deceleration parameter $q(t)$ depend on a hidden parameter ε and its derivative $\dot\varepsilon$. For the universe having the flat curvature one can infer the following relation between pressure and density parameters:

There are 0*-functions* ξ, ζ *such that* $p = \hat{w}\rho c^2$*, where* $\hat{w}(t) = w - t\dot{\xi} + \zeta$*.*

We see that the deceleration parameter $q(t)$, the equation of state and the pressure $p(t)$ contain not only the "hidden" parameter $\varepsilon(t)$, but $\dot\varepsilon(t)$ and $t\dot\varepsilon(t)$ as well.

While $\varepsilon(t)$ is a 0-function, $\dot\varepsilon(t)$ and $t\dot\varepsilon(t)$ need not to be. In fact they can be unbounded and oscillatory as well. Examples of this kind are:

$\varepsilon(t) = \cos(t^2)/t, \quad \varepsilon(t) = \sin(t^3)/t.$

It means that $q(t)$ and $p(t)$ can be unbounded and oscillatory as well. It seems that this fact is overlooked in the classical cosmology, mainly due to the absence of "microscopic" analysis which give us the theory of regularly varying functions.

Therefore, it is of interest to describe in more detail the hidden parameter ε. We found that ε is a solution of the Riccati differential equation:

$$t\dot\varepsilon(t) = (1 - 2\alpha)\varepsilon - \varepsilon^2 + \alpha(1-\alpha) - \mu(t). \tag{15}$$

where $\mathbf{M}(\mu) = \alpha(1-\alpha)$ and ε is a 0-function. If $\lim_{t\to\infty} \mu(t) = \alpha(1-\alpha)$, then $t\dot\varepsilon$ is a 0-function, hence $\dot\varepsilon$ and $t\dot\varepsilon$ can be neglected in the representation of cosmological parameters. In this case $q(t)$ and equation of state reduce to their standard form in classical cosmology:

$$q(t) = \frac{1+3w}{2}, \quad p(t) = wc^2\rho(t)$$

where w is a constant (c is the the speed of light). Hence, the only interesting case would be when $\mathbf{M}(q) = \alpha(1-\alpha)$ and $\lim_{t\to\infty} \mu(t)$ does not exist. Then the "hidden" parameter ε, in fact its derivative $\dot\varepsilon$, might have the strong influence in the asymptotical behavior of the parameters $q(t)$ and $p(t)$.

We note that J. Barrow (see, for example, J. Barrow, D. Shaw, 2008) also discussed the asymptotic behavior of cosmological parameters based on the theory of Hardy fields. We note that this theory precedes Karamata's theory of regular variation.

Several physical models can be proposed taking into account the "hidden" parameter $\varepsilon(t)$:

1. Dark matter and dark energy are in the equilibrium but small fluctuation in this state produce variation of $q(t)$ and $p(t)$.

2. Variations of $q(t)$ and $p(t)$ are the consequences of the echo effect which appeared in the inflationary epoch which lasted from 10^{-36} seconds after the Big Bang until 10^{-32} seconds (Alan Guth and Andrei Linde, 1981).

3. Variations of $q(t)$ and $p(t)$ are the consequences of the existence and the influence of the dual universe.

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