DUAL PHOTONS AND GRAVITONS

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Abstract. We review the status of electric/magnetic duality for free gauge field theories in four space-time dimensions with emphasis on Maxwell theory and linearized Einstein gravity. Using the theory of vector and tensor spherical harmonics, we provide explicit construction of dual photons and gravitons by decomposing the fields into axial and polar configurations with opposite parity and interchanging the two sectors. When the theories are defined on AdS_4 space-time there are boundary manifestations of the duality, which for the case of gravity account for the energy-momentum/Cotton tensor duality (also known as dual graviton correspondence). For AdS_4 black-hole backgrounds there is no direct analogue of gravitational duality on the bulk, but there is still a boundary duality for quasi-normal modes satisfying a selected set of boundary conditions. Possible extensions of this framework and some open questions are also briefly discussed.

1. INTRODUCTION

Electric/magnetic duality has been the cornerstone for many developments in theoretical and mathematical physics in the past fifteen years. It has its roots in Dirac's original work on electromagnetism, but it certainly goes well beyond it in many ways. In recent years it has also been generalized to other free gauge field theories including linearized gravity in four space-time dimensions. Some implications of electric/magnetic duality have also been studied in the context of AdS_4/CFT_3 correspondence, while several other ideas related to duality in gravitational theories are still waiting to find their proper place in this framework and (hopefully) turn into powerful tools.

The purpose of these lecture notes is to review the basic elements of duality in electromagnetism and gravity and resolve the non-local relations that define the dual field variables in terms of the original ones. This is practically achieved by employing group theory techniques for solving wave equations on spherically symmetric backgrounds. The main emphasis here is on the methods that lead to the resolution of dualities, whereas the actual details of the calculations are omitted in some cases to avoid lengthy formulae; the interested reader will be referred to the literature for further technicalities. We found it rather instructive to present these methods in parallel for Maxwell theory and linearized gravity because of the common aspects they share. Our treatment of the problem for Maxwell equations is quite elementary, but is not usually found elsewhere. Thus, we include it here for completeness and in order to motivate in pedagogical way similar constructions for linearized gravity. As application of all these results, we discuss the boundary manifestation of electric/magnetic

duality for theories defined on AdS_4 space-time and explain, in particular, how the dual graviton correspondence comes into play in holography. Lack of space does not allow us to expand further on these applications; here, we only give a flavor of the additional features one has to account and understand better in the context of AdS_4/CFT_3 correspondence.

Thus, the material of this contribution is organized as follows. Section 2 provides a brief outline of the electric/magnetic duality in Maxwell theory and linearized gravity together with the definition of dual photons and gravitons. Section 3 contains some background mathematical material from the theory of spherical harmonics, which are used to decompose gauge fields as well as metric perturbations into the axial and polar sectors with opposite parity. Section 4 makes use of the classical equations of motion, thus reducing the corresponding wave equations into an effective Schrödinger problem. Then, duality is resolved group theoretically by exchanging the axial and polar sectors of the theories and leads to explicit construction of the dual photons and gravitons. Section 5 makes use of holographic renormalization to examine the boundary manifestation of electric/magnetic duality on AdS_4 space-time for both Maxwell theory and linearized gravity. Certain generalizations to AdS_4 black-hole backgrounds are also briefly discussed in this context. Finally, section 6 contains our conclusions and discusses possible extensions of the formalism to address some open problems.

Further details can be found in the published works (see Bakas 2009ab) and references therein.

2. ELECTRIC/MAGNETIC DUALITY

Free gauge fields in four space-time dimensions exhibit two physical degrees of freedom that can be rotated into one another by a canonical transformation mixing the two pairs of unconstrained dynamical variables, while keeping the Hamiltonian form-invariant. It is a general result that extends the electric/magnetic duality of electromagnetism to other physical fields including Einstein gravity. In this section we briefly review the status of duality in Maxwell and Einstein theories and provide the definition of dual photons and dual gravitons. Their explicit construction will be described in detail in subsequent sections.

2. 1. MAXWELL THEORY

Let us consider the source free Maxwell theory on a fixed (generally curved) space-time manifold M_4 described in terms of a $U(1)$ gauge field A_μ. The field strength is defined as usual,

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu \ , \tag{1}$$

and the dual field strength as

$$^\star F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu}{}^{\kappa\lambda}F_{\kappa\lambda} \tag{2}$$

using the covariant fully antisymmetric symbol on M_4.

The field equations and the Bianchi identities assume the familiar form

$$\nabla_\nu F^{\mu\nu} = 0 = \nabla_\nu {}^\star F^{\mu\nu} \tag{3}$$

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and their role is interchanged under the operation of duality satisfying the identity $\star^2 = -1$. When expressed in terms of physical fields,

$$E_a = F_{ta} , \qquad B_a = {}^\star F_{ta} \tag{4}$$

it interchanges electric and magnetic components as follows,

$$E_a \to B_a , \qquad B_a \to -E_a , \tag{5}$$

thus giving rise to the celebrated electric/magnetic duality of Maxwell theory.

Then, for a given gauge field A_μ, the dual configuration $\tilde{A}_\mu$ is defined by the equations

$$^\star F_{\mu\nu}(A) = F_{\mu\nu}(\tilde{A}) , \tag{6}$$

$$^\star F_{\mu\nu}(\tilde{A}) = -F_{\mu\nu}(A) . \tag{7}$$

$\tilde{A}_\mu$, which we call it the dual photon, is uniquely defined, up to gauge transformations, but it is non-locally related to the original field A_μ. It is our purpose to provide explicit construction of the dual field in a broad class of static spherically symmetric backgrounds M_4. It will also prove a useful guide for performing similar constructions in linearized gravity around (a more restricted class of) spherically symmetric backgrounds.

We also note for completeness that one may consider an $SO(2)$ rotation of the electric and magnetic fields parametrized by an arbitrary angle δ,

$$\begin{pmatrix} E'_a \\ B'_a \end{pmatrix} = \begin{pmatrix} \cos\delta & \sin\delta \\ -\sin\delta & \cos\delta \end{pmatrix} \begin{pmatrix} E_a \\ B_a \end{pmatrix}, \tag{8}$$

thus providing a more general action on the space of solutions of Maxwell theory.

2. 2. LINEARIZED GRAVITY

For gravity, the electric/magnetic duality is only defined at the linear level by considering small perturbations around a reference metric,

$$g_{\mu\nu} = g_{\mu\nu}^{(0)} + h_{\mu\nu} . \tag{9}$$

Then, duality is described as a non-local transformation among perturbations of the same reference metric

$$\tilde{g}_{\mu\nu} = g_{\mu\nu}^{(0)} + \tilde{h}_{\mu\nu} , \tag{10}$$

which acts on the space of solutions of vacuum Einstein equation and which can also be realized as symmetry of the gravitational action; the duality breaks down at the first self-interacting cubic approximation to general relativity, but this will not be relevant to the present work. The actual differential equations that define the dual graviton $\tilde{h}_{\mu\nu}$ in terms of $h_{\mu\nu}$ resemble those of Maxwell theory and will be written down shortly.

We mention right from the beginning that the reference metric $g_{\mu\nu}^{(0)}$ cannot be arbitrary, and, hence, gravitational duality (as known to this day) is only restricted to

small perturbations around special backgrounds. These are provided by the metric of flat Minkowski space-time when the cosmological constant $\Lambda = 0$ and by the metric of $(A)dS_4$ space-time when $\Lambda \neq 0$. The reason for it, as will emerge from the formalism, is that these special backgrounds are trivially self-dual (in an appropriate sense) and they remain inert under duality. Then, one is only left to compare the fluctuations $h_{\mu\nu}$ and establish the action of the duality symmetry.

The electric/magnetic duality of linearized gravity was initially formulated for $\Lambda = 0$, but it extends rather easily to vacuum Einstein equations with cosmological constant (see Nieto 1999, Hull 2001, Henneaux & Teitelboim 2005, Deser & Seminara 2005, Julia et al. 2005, Leigh & Petkou 2007, Bakas 2009a). Here, following Bakas 2009a, we formulate directly the problem for arbitrary Λ using the quantity

$$Z_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{\Lambda}{3} \left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho} \right) \ . \tag{11}$$

$Z_{\mu\nu\rho\sigma}$ arises by restricting the Weyl curvature tensor in four space-time dimensions to on-shell metrics, and, clearly, it fulfills the identities

$$Z_{\mu[\nu\rho\sigma]} = 0 \ , \qquad \nabla_{[\lambda}Z_{\mu\nu]\rho\sigma} = 0 \ , \tag{12}$$

and the on-shell metrics satisfy the equation

$$Z^{\rho}{}_{\mu\rho\nu} \equiv Z_{\mu\nu} = 0 \ , \tag{13}$$

which is equivalent to Einstein equations with cosmological constant Λ, i.e., $R_{\mu\nu} = \Lambda g_{\mu\nu}$. One also defines the dual curvature tensor

$$^{\star}Z_{\mu\nu\rho\sigma} = \frac{1}{2}\epsilon_{\mu\nu}{}^{\kappa\lambda}Z_{\kappa\lambda\rho\sigma} \ , \tag{14}$$

which fulfills similar identities, but with reverse meaning.

Linearized gravity around $g_{\mu\nu}^{(0)}$ exhibits a duality that exchanges Bianchi identities with the classical equations of motion, as in electromagnetism, by letting

$$^{\star}Z_{\mu\nu\rho\sigma}(g) = Z_{\mu\nu\rho\sigma}(\tilde{g}) \ , \tag{15}$$

$$^{\star}Z_{\mu\nu\rho\sigma}(\tilde{g}) = -Z_{\mu\nu\rho\sigma}(g) \ . \tag{16}$$

Actually, at the linear level, it is appropriate to replace the covariant derivatives ∇ by ordinary derivatives ∂ and work out the linear differential equations that uniquely define $\tilde{h}_{\mu\nu}$ in terms of $h_{\mu\nu}$, up to reparametrizations. This is in exact analogy with electromagnetism.

Also, it is useful to introduce the electric and magnetic components of the Weyl tensor as

$$E_{ab} = Z_{atbt} \ , \qquad B_{ab} = {}^{\star}Z_{atbt} \ . \tag{17}$$

Then, the gravitational duality transformation is realized as

$$E_{ab} \rightarrow B_{ab} \ , \qquad B_{ab} \rightarrow -E_{ab} \ . \tag{18}$$

The electric and magnetic tensors are represented on shell by 3×3 symmetric traceless matrices, and, as such, they have five independent components each. Furthermore,

the background metric $g_{\mu\nu}^{(0)}$, which is used to exhibit duality for the appropriate choice of Λ, is trivially self-dual, since it has $E_{ab} = 0 = B_{ab}$.

More generally, as for electromagnetism, one may consider an $SO(2)$ rotation of these components parametrized by an arbitrary angle δ,

$$\begin{pmatrix} E'_{ab} \\ B'_{ab} \end{pmatrix} = \begin{pmatrix} \cos\delta & \sin\delta \\ -\sin\delta & \cos\delta \end{pmatrix} \begin{pmatrix} E_{ab} \\ B_{ab} \end{pmatrix}, \tag{19}$$

thus providing a more general action on the space of solutions of linearized gravity. Details of the proof can be found in the original works (see, in particular, Henneaux & Teitelboim 2005 and Julia et al. 2005).

Our purpose in the sequel is to provide explicit construction of the dual gravitons, in analogy with the dual photons, by relying on group theory techniques for solving wave equations on spherically symmetric space-times.

3. ANALYSIS INTO SPHERICAL HARMONICS

We consider static spherically symmetric space-times M_4 with local coordinates $(t,\ r,\ \theta,\ \phi)$ and line element

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\ . \tag{20}$$

Wave equations for physical fields, such as a gauge field A_μ satisfying the source-free Maxwell equations on M_4, and a graviton field $h_{\mu\nu}$ satisfying the linearized Einstein equations for metric perturbations around M_4, can be reduced to a radial (effective) Schrödinger equation by assuming factorization of their solutions on the $(t,\ r)$ and $(\theta,\ \phi)$ coordinates. In this section, we present the background mathematical material to achieve this purpose for fields defined on M_4.

3. 1. GENERALITIES

The dependence of fields upon θ and ϕ is solely described by group theory using spherical harmonics and generalizations thereof. In particular, one should consider the theory of spherical harmonics for vector and tensor fields, generalizing the usual spherical harmonics $Y_l^m(\theta, \phi)$ for scalar fields, to higher rank configurations.

The spherical symmetry of M_4 allows angular momentum to be defined and investigated by studying rotations on the two-dimensional manifold having $t = $ constant and $r = $ constant. Thus, to set up the notation, we consider the sphere S^2 with local coordinates $(\theta,\ \phi)$ and metric tensor

$$\gamma_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}. \tag{21}$$

We also define the covariant fully antisymmetric symbol on S^2 with non-vanishing components

$$\epsilon^{\theta\phi} = -\epsilon^{\phi\theta} = \frac{1}{\sin\theta}\ . \tag{22}$$

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Under rigid rotations around the origin, the various components of physical fields defined on M_4 are decomposed into spherical harmonics by viewing them as covariant quantities on the sphere S^2. Then, the theory of generalized spherical harmonics (see, for instance, Regge & Wheeler 1957, Zerilli 1970a and the review article Thorne 1980) can be tabulated as follows, for scalar, vector and rank-two tensor fields defined on S^2, all having definite parity:

Scalar: They are simply described by ordinary spherical harmonics with $l = 0, 1, 2, \cdots$ and $-l \leq m \leq l$,

$$Y_l^m \ , \qquad \text{with parity } (-1)^l \ . \tag{23}$$

Vectors: There are two different type of vector spherical harmonics with opposite parity, namely

$$(\Psi_l^m)_i \ = \ \partial_i Y_l^m \ , \qquad \text{with parity } (-1)^l \ , \tag{24}$$

$$(\Phi_l^m)_i \ = \ \epsilon_i{}^j \partial_j Y_l^m \ , \qquad \text{with parity } (-1)^{l+1} \ . \tag{25}$$

Rank-two tensors: In this case there are three different type of tensor spherical harmonics with the following parities,

$$(\Psi_l^m)_{ij} \ = \ \nabla_i \nabla_j Y_l^m \ , \qquad \text{with parity } (-1)^l \ , \tag{26}$$

$$(\Phi_l^m)_{ij} \ = \ \gamma_{ij} Y_l^m \ , \qquad \text{with parity } (-1)^l \ , \tag{27}$$

$$(X_l^m)_{ij} \ = \ \frac{1}{2}\Big[\epsilon_i{}^k (\Psi_l^m)_{kj} + \epsilon_j{}^k (\Psi_l^m)_{ki} \Big] \ , \quad \text{with parity } (-1)^{l+1} \ . \tag{28}$$

In the examples of gauge fields A_μ and rank-two tensor fields $h_{\mu\nu}$ on M_4 that will be considered below in detail, all scalar, vector and tensor harmonics come naturally into play. In all cases, it is particularly convenient to separate the fields into two distinct classes with definite, but opposite parity, since the general configuration is a linear superposition of them. The *polar* class corresponds to fields of parity $(-1)^l$, whereas the *axial* class to parity $(-1)^{l+1}$. They can be though as the "electric" and "magnetic" parts of the fields, respectively, due to their parity, and, as such, they provide the right decomposition to realize electric-magnetic duality in a natural way. Clearly, this decomposition is invariant under rigid rotations, since the parity does not change.

We also note that the radial equations that result from wave equations in spherically symmetric space-times depend only on the angular momentum quantum number l and not on m, as in the hydrogen atom. Thus, without loss of generality, we may drop the ϕ dependence by specializing to axisymmetric field configurations with $m = 0$, so that $Y_l^m(\theta, \phi)$ reduce to $P_l(\cos\theta)$, to simplify the derivation of the radial equations described in the subsequent sections. This will be discussed further shortly, following the general harmonic decomposition of the fields.

One may similarly consider the harmonic decomposition of higher rank tensor fields, but such generalization will not be in focus in the present work, since we only consider the theories of photons and gravitons.

3. 2. GAUGE FIELD DECOMPOSITION

Under rigid rotations around the origin, the four components of a gauge field A_μ on M_4 transform like two scalars A_t, A_r and one vector (A_θ, A_ϕ), when considered as

covariant quantities on the sphere S^2. Thus, it is natural to use a $2+2$ splitting of the space-time components of A_μ, as

$$A_\mu(t, r, \theta, \phi) = \begin{pmatrix} S \\ V \end{pmatrix}, \qquad (29)$$

denoting by (S) the scalars and by (V) the vector. Then, any A_μ can be decomposed into two distinct classes of opposite parity with respect to angular momentum (see, for instance, Ruffini et al. 1972):

Axial gauge field: For gauge field configurations of parity $(-1)^{l+1}$ the two scalars obviously vanish,

$$S_\text{axial} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad (30)$$

and the vector part is dictated by the form of the corresponding vector spherical harmonic $(\Phi_l^m)_i$, using an arbitrary multiplicative function $a(t, r)$, as

$$V_\text{axial} = a(t, r) \begin{pmatrix} -\frac{1}{\sin\theta} \frac{\partial}{\partial\phi} \\ \sin\theta \frac{\partial}{\partial\theta} \end{pmatrix} Y_l^m(\theta, \phi) . \qquad (31)$$

For axisymmetric configurations, with $m = 0$, the most general gauge field with parity $(-1)^{l+1}$ takes the form

$$A_\mu^\text{axial} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ a(r) \end{pmatrix} e^{-i\omega t} \sin\theta \, \partial_\theta P_l(\cos\theta) \qquad (32)$$

by also factorizing the t and r dependence of the function $a(t, r)$.

Polar gauge field: For gauge field configurations of parity $(-1)^l$ the scalar part is parametrized in terms of two arbitrary functions $C(t, r)$ and $D(t, r)$ as

$$S_\text{polar} = \begin{pmatrix} C(t, r) \\ D(t, r) \end{pmatrix} Y_l^m(\theta, \phi), \qquad (33)$$

whereas the vector part is expressed in terms of the corresponding vector spherical harmonic $(\Psi_l^m)_i$ as

$$V_\text{polar} = b(t, r) \begin{pmatrix} \frac{\partial}{\partial\theta} \\ \frac{\partial}{\partial\phi} \end{pmatrix} Y_l^m(\theta, \phi) , \qquad (34)$$

using an arbitrary function $b(t, r)$. Note, however, that the vector part can always be gauged away to zero by the transformation

$$A_\mu' = A_\mu + \partial_\mu \epsilon \qquad (35)$$

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choosing, in particular, the parameter

$$\epsilon = -b(r,t)Y_l^m(\theta,\phi) \ . \tag{36}$$

Thus, only the scalar part is non-vanishing in this case.

For axisymmetric configurations, with $m = 0$, the most general gauge field with parity $(-1)^l$ takes the form

$$A_\mu^{\text{polar}} = \begin{pmatrix} C(r) \\ D(r) \\ 0 \\ 0 \end{pmatrix} e^{-i\omega t} P_l(\cos\theta) \tag{37}$$

by also factorizing the t and r dependence of the functions $C(t,r)$ and $D(t,r)$.

In the sequel, we assume without lose of generality that the gauge fields A_μ are decomposed invariantly into axial and polar parts given by the axisymmetric expressions (32) and (37). The equations for the corresponding radial functions $a(r)$ and $C(r)$, $D(r)$ follow from the Maxwell equations in vacuum, as will be seen later.

3. 3.   METRIC DECOMPOSITION

Under rigid rotations around the origin, the ten components of the tensor field $h_{\mu\nu}$ on M_4 transform like three scalars h_{tt}, h_{tr}, h_{rr}, two vectors $(h_{t\theta}, h_{t,\phi})$ and $(h_{r\theta}, h_{r\phi})$, and a rank-two tensor (accounting for the remaining components), when considered as covariant quantities on the sphere S^2. Thus, it is natural to use a $2+2$ splitting of the symmetric matrix $h_{\mu\nu}$ into blocks

$$h_{\mu\nu}(t,r,\theta,\phi) = \begin{pmatrix} S & V \\ \tilde{V} & T \end{pmatrix} \tag{38}$$

denoting by (S) the scalars by (V) the vectors and by (T) the tensor, whereas $\tilde{V}$ denotes the transpose of the 2×2 matrix V. Then, any $h_{\mu\nu}$ can be decomposed into two distinct classes of opposite parity with respect to angular momentum (see Regge & Wheeler 1957):

Axial metric perturbations: For metric perturbations of parity $(-1)^{l+1}$ the scalar part obviously vanishes,

$$S_{\text{axial}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \ . \tag{39}$$

The vector part contains two vector spherical harmonics $(\Phi_l^m)_i$ with parity $(-1)^{l+1}$, each one multiplied with an arbitrary function $h_0(t,r)$ and $h_1(t,r)$, and takes the form

$$V_{\text{axial}} = \begin{pmatrix} -h_0(t,r)\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} & h_0(t,r)\sin\theta\frac{\partial}{\partial\theta} \\ & \\ -h_1(t,r)\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} & h_1(t,r)\sin\theta\frac{\partial}{\partial\theta} \end{pmatrix} Y_l^m(\theta,\phi) \ . \tag{40}$$

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Finally, the tensor part is obtained from the corresponding rank-two tensor spherical harmonic $(X_l^m)_{ij}$ by multiplying it with an arbitrary function $h_2(t,r)$,

$$
T_{\text{axial}} = \frac{1}{2}
\begin{pmatrix}
2\left(\frac{1}{\sin\theta}\frac{\partial^2}{\partial\theta\partial\phi} - \frac{\cos\theta}{\sin^2\theta}\frac{\partial}{\partial\phi}\right) & \frac{1}{\sin\theta}\frac{\partial^2}{\partial\phi^2} - \sin\theta\frac{\partial^2}{\partial\theta^2} + \cos\theta\frac{\partial}{\partial\theta} \\[2ex]
\frac{1}{\sin\theta}\frac{\partial^2}{\partial\phi^2} - \sin\theta\frac{\partial^2}{\partial\theta^2} + \cos\theta\frac{\partial}{\partial\theta} & -2\left(\sin\theta\frac{\partial^2}{\partial\theta\partial\phi} - \cos\theta\frac{\partial}{\partial\phi}\right)
\end{pmatrix}
$$
$$
\times h_2(t,r)Y_l^m(\theta,\phi) \ . \tag{41}
$$

Note, however, that one may employ reparametrizations generated by a vector field ξ_μ,

$$
h'_{\mu\nu} = h_{\mu\nu} + \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu \ , \tag{42}
$$

in order to simplify the expressions. For example, the terms containing second order derivatives in θ and ϕ can be gauged away using

$$
\xi_\mu = \Lambda(t,r)\left(0,\ 0,\ -\frac{1}{\sin\theta}\frac{\partial}{\partial\phi},\ \sin\theta\frac{\partial}{\partial\theta}\right)Y_l^m(\theta,\phi) \tag{43}
$$

with appropriately chosen function $\Lambda(t,r)$ to also annul $h_2(t,r)$; this ξ_μ has exactly the same form as an axial gauge field with parity $(-1)^{l+1}$.

For axisymmetric configurations, with $m = 0$, the general form of metric perturbations with parity $(-1)^{l+1}$ becomes, after taking into account the freedom of reparametrizations,

$$
h_{\mu\nu}^{\text{axial}} =
\begin{pmatrix}
0 & 0 & 0 & h_0(r) \\[1ex]
0 & 0 & 0 & h_1(r) \\[1ex]
0 & 0 & 0 & 0 \\[1ex]
h_0(r) & h_1(r) & 0 & 0
\end{pmatrix}
e^{-i\omega t}\sin\theta\ \partial_\theta P_l(\cos\theta) \tag{44}
$$

by also factorizing the t and r dependence of the functions $h_0(t,r)$ and $h_1(t,r)$.

Polar metric perturbations: For metric perturbations of parity $(-1)^l$ the scalar part is parametrized by 3 arbitrary functions $H_0(t,r)$, $H_1(t,r)$ and $H_2(t,r)$, one for each scalar,

$$
S_{\text{polar}} =
\begin{pmatrix}
f(r)H_0(t,r) & H_1(t,r) \\[2ex]
H_1(t,r) & H_2(t,r)/f(r)
\end{pmatrix}
Y_l^m(\theta,\phi) \ . \tag{45}
$$

The vector part is described by two vector spherical harmonics $(\Psi_l^m)_i$ of the corresponding parity, using two arbitrary multiplicative functions $h_0(t,r)$ and $h_1(t,r)$,

$$
V_{\text{polar}} =
\begin{pmatrix}
h_0(t,r)\frac{\partial}{\partial\theta} & h_0(t,r)\frac{\partial}{\partial\phi} \\[2ex]
h_1(t,r)\frac{\partial}{\partial\theta} & h_1(t,r)\frac{\partial}{\partial\phi}
\end{pmatrix}
Y_l^m(\theta,\phi) \ . \tag{46}
$$

Finally, the tensor part is composed out of two rank-two tensor spherical harmonics $(\Psi_l^m)_{ij}$ and $(\Phi_l^m)_{ij}$ of parity $(-1)^l$ with arbitrary coefficients $G(t,r)$ and $K(t,r)$, respectively, and it takes the following form,

$$
T_{\text{polar}} = r^2 G(t,r)
\begin{pmatrix}
\frac{\partial^2}{\partial\theta^2} & \frac{\partial^2}{\partial\theta\partial\phi} - \cot\theta\frac{\partial}{\partial\phi} \\
\\
\frac{\partial^2}{\partial\theta\partial\phi} - \cot\theta\frac{\partial}{\partial\phi} & \frac{\partial^2}{\partial\phi^2} + \sin\theta\cos\theta\frac{\partial}{\partial\theta}
\end{pmatrix}
Y_l^m(\theta,\phi)
$$

$$
+ r^2 K(t,r)
\begin{pmatrix}
1 & 0 \\
\\
0 & \sin^2\theta
\end{pmatrix}
Y_l^m(\theta,\phi) .
\tag{47}
$$

As before, there is also the freedom of making reparametrizations generated by a vector field ξ_μ,

$$
h'_{\mu\nu} = h_{\mu\nu} + \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu ,
\tag{48}
$$

which can be chosen to gauge away the terms containing second derivatives in θ and ϕ. A particularly useful choice that serves this purpose is provided by

$$
\xi_\mu = \left(M_0(t,r),\ M_1(t,r),\ M(t,r)\frac{\partial}{\partial\theta},\ M(t,r)\frac{\partial}{\partial\phi} \right) Y_l^m(\theta,\phi)
\tag{49}
$$

with appropriately chosen functions $M_0(t,r)$, $M_1(t,r)$ and $M(t,r)$ to also annul $h_0(t,r)$, $h_1(t,r)$ and $G(t,r)$; here, ξ_μ has exactly the same form as a polar gauge field with parity $(-1)^l$.

For axisymmetric configurations, with $m = 0$, the general form of metric perturbations with parity $(-1)^l$ becomes, after taking into account the freedom of reparametrizations,

$$
h_{\mu\nu}^{\text{polar}} =
\begin{pmatrix}
f(r)H_0(r) & H_1(r) & 0 & 0 \\
\\
H_1(r) & H_2(r)/f(r) & 0 & 0 \\
\\
0 & 0 & r^2 K(r) & 0 \\
\\
0 & 0 & 0 & r^2 K(r)\sin^2\theta
\end{pmatrix}
e^{-i\omega t} P_l(\cos\theta)
\tag{50}
$$

by also factorizing the t and r dependence of the four functions $H_0(t,r)$, $H_1(t,r)$, $H_2(t,r)$ and $K(t,r)$.

In the following, we assume without lose of generality that the metric perturbations $h_{\mu\nu}$ are decomposed invariantly into axial and polar parts given by the axisymmetric expressions (44) and (50). The equations for the corresponding radial functions $h_0(r)$, $h_1(r)$ and $H_0(r)$, $H_1(r)$, $H_2(r)$, $K(r)$ follow from the linearized Einstein equations in vacuum, as will be seen later.

4. RESOLVING THE DUALITY RELATIONS

In this section we impose the field equations and reduce Maxwell theory as well as linear gravity to an effective Schrödinger problem, as for all wave equations on spherically symmetric backgrounds (see also Ishibashi & Wald 2004 for a comprehensive but general discussion). The main result that will be described here is the manifestation of electric/magnetic duality as axial/polar interchange among field configurations with opposite parity, following Bakas 2009a. This resolves the duality relations and provides explicit construction of the dual photons and gravitons, in close analogy to each other, by group theory methods. The results follow by direct computation without the need to construct explicit solutions of the Schrödinger equation; these solutions will only be presented later in the context of holographic computations in AdS_4 space-time.

4. 1. DUAL PHOTONS

First, it can be shown that Maxwell equations on a spherically symmetric background with metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \ , \tag{51}$$

with arbitrary $f(r)$, are reduced to an effective Schrödinger problem (see Ruffini et al. 1972, Ishibashi & Wald 2004),

$$\left(-\frac{d^2}{dr_\star^2} + f(r)\frac{l(l+1)}{r^2}\right)\Psi(r) = \omega^2\Psi(r) \tag{52}$$

with respect to the tortoise radial coordinate $r_\star$, which is defined as follows,

$$dr_\star = \frac{dr}{f(r)} \ . \tag{53}$$

The demonstration will be made separately for the axial and polar sectors using the decomposition of the gauge field into spherical harmonics of given parity. The effective wave function will also be identified with appropriate gauge invariant components of the field A_μ, so that electric/magnetic duality can be readily implemented by exchanging axial with polar sectors. This way, the explicit construction of $\tilde{A}_\mu$ will follow from A_μ up to gauge transformations.

Axial sector: Using the form of the gauge potentials (32), let us first compute the non-vanishing components of the field strength which take the form

$$
\begin{align}
F_{t\phi} &= -i\omega a_l(r)e^{-i\omega t}\sin\theta \ \partial_\theta P_l(\cos\theta) \ , \tag{54}\\
F_{r\phi} &= a_l'(r)e^{-i\omega t}\sin\theta \ \partial_\theta P_l(\cos\theta) \ , \tag{55}\\
F_{\theta\phi} &= -l(l+1)a_l(r)e^{-i\omega t}\sin\theta \ P_l(\cos\theta) \ . \tag{56}
\end{align}
$$

In terms of physical electric and magnetic fields one sees that the axial sector has non-vanishing E_ϕ, B_r and B_θ.

It follows by straightforward calculation that Maxwell equations $\nabla_\nu F^{\mu\nu} = 0$ reduce to the Schrödinger equation (52) with the identification of wave-function,

$$\Psi_{\text{axial}}(r) = a_l(r) \ . \tag{57}$$

<u>Polar sector</u>: In this case, using the form of the gauge potentials (37), the non-vanishing components of the field strength turn out to be

$$F_{tr} = -\left(C'_l(r) + i\omega D_l(r)\right)e^{-i\omega t}P_l(\cos\theta) , \tag{58}$$

$$F_{t\theta} = -C_l(r)e^{-i\omega t}\partial_\theta P_l(\cos\theta) , \tag{59}$$

$$F_{r\theta} = -D_l(r)e^{-i\omega t}\partial_\theta P_l(\cos\theta) . \tag{60}$$

The non-vanishing components of the physical electric and magnetic fields in the polar sector are E_r, E_θ and B_ϕ, which are complementary to those of the axial sector; it will shortly be seen that this is not an accident.

As before, Maxwell equations reduce to the Schrödinger equation (52) provided that the following identification is made for the wave-function

$$\Psi_{\text{polar}}(r) = r^2\left(C'_l(r) + i\omega D_l(r)\right) . \tag{61}$$

Actually, it also turns out that the non-trivial components of the polar gauge field (37) are expressed in terms of the corresponding wave-function as

$$C_l(r) = \frac{1}{l(l+1)}\frac{d}{dr_\star}\Psi_{\text{polar}}(r) , \tag{62}$$

$$D_l(r) = -\frac{i\omega}{l(l+1)f(r)}\Psi_{\text{polar}}(r) . \tag{63}$$

<u>Duality relation</u>: It is a simple matter to verify that the two sectors are mutually related by electric/magnetic duality, i.e.,

$$^\star F_{\mu\nu}^{\text{axial}} = F_{\mu\nu}^{\text{polar}} , \tag{64}$$

$$^\star F_{\mu\nu}^{\text{polar}} = -F_{\mu\nu}^{\text{axial}} , \tag{65}$$

provided that the same boundary conditions are imposed on the axial and polar wave-functions so that they are the same (up to an arbitrary factor) and the frequencies ω are also the same. With the normalization chosen above, matching is exact for

$$\Psi_{\text{polar}} = l(l+1)\Psi_{\text{axial}}(r) . \tag{66}$$

Thus, axial/polar interchange provides explicit construction of the dual photons as advertised above,

$$\tilde{A}_\mu^{\text{axial}} = A_\mu^{\text{polar}} , \qquad \tilde{A}_\mu^{\text{polar}} = A_\mu^{\text{axial}} . \tag{67}$$

This result will be used to guide a similar construction for the dual gravitons.

4. 2. DUAL GRAVITONS

Next, we examine the linearized Einstein equations on a restricted class of spherically symmetric backgrounds

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2) , \tag{68}$$

with cosmological constant Λ and profile function

$$f(r) = 1 - \frac{\Lambda}{3}r^2 \ . \tag{69}$$

The field equations $\delta R_{\mu\nu} = \Lambda h_{\mu\nu}$ can also be reduced to an effective Schrödinger problem (see, for instance, Ishibashi & Wald 2004 and Bakas 2009a)

$$\left(-\frac{d^2}{dr_\star^2} + f(r)\frac{l(l+1)}{r^2}\right)\Psi(r) = \omega^2\Psi(r) \tag{70}$$

with respect to the tortoise radial coordinate $r_\star$. Note that the effective potential is the same as in Maxwell theory.

Axial sector: Using the form of the metric perturbations (44) we find that the field equations reduce to the following system of first order differential equations for the two unknown radial functions $h_0(r)$ and $h_1(r)$,

$$\frac{2}{r}h_0(r) - h_0'(r) = i\frac{f(r)}{\omega}\left(\frac{\omega^2}{f(r)} - \frac{(l-1)(l+2)}{r^2}\right)h_1(r) \ , \tag{71}$$

$$h_0(r) = i\frac{f(r)}{\omega}\left(f(r)h_1(r)\right)' \ . \tag{72}$$

These equations give rise to the effective Schrödinger equation (70) with wave-function

$$\Psi_{\mathrm{axial}}(r) = \frac{f(r)}{r}h_1(r) \ . \tag{73}$$

Clearly, $h_0(r)$ is also expressed in terms of $\Psi_{\mathrm{axial}}(r)$ as

$$h_0(r) = \frac{i}{\omega}\frac{d}{dr_\star}(r\Psi_{\mathrm{axial}}(r)) \ . \tag{74}$$

Polar sector: In this case, the metric perturbations assume the form (50), which depends on four unknown radial functions $H_0(r)$, $H_1(r)$, $H_2(r)$ and $K(r)$. The field equations yield the following relation

$$H_0(r) = H_2(r) \ , \tag{75}$$

as well as the system of first order equations

$$rK'(r) + \frac{K(r)}{f(r)} - H_0(r) - i\frac{l(l+1)}{2\omega r}H_1(r) = 0 \ , \tag{76}$$

$$(f(r)H_0(r))' - f(r)K'(r) + i\omega H_1(r) = 0 \ , \tag{77}$$

$$(f(r)H_1(r))' + i\omega(H_0(r) + K(r)) = 0 \ . \tag{78}$$

Furthermore, consistency of the remaining second order field equations requires an algebraic relation among the remaining three functions,

$$(l-1)(l+2)H_0(r) - \frac{2ir}{\omega}\left(\omega^2 + \frac{\Lambda}{6}l(l+1)\right)H_1(r) =$$

$$= \left(l(l+1) - \frac{2}{f(r)}(\omega^2 r^2 + 1)\right)K(r) \ , \tag{79}$$

which can be regarded as first integral of the first order system above.

It can be verified that these coupled differential equations give rise to the same Schrödinger problem (70), as before, provided that the effective wave-function is constructed as follows,

$$\Psi_{\text{polar}}(r) = \frac{2r}{(l-1)(l+2)} \left(K(r) - i \frac{f(r)}{\omega r} H_1(r) \right) . \tag{80}$$

Actually, all radial functions can be written in terms of $\Psi_{\text{polar}}(r)$ and its derivative, but the explicit expressions are more involved now. They read as

$$H_0(r) = H_2(r) = \left(\frac{l(l+1)}{2r} - \frac{\omega^2 r}{f(r)} + \frac{d}{dr} \right) \Psi_{\text{polar}}(r) , \tag{81}$$

$$H_1(r) = -\frac{i\omega}{f(r)} \left(1 + r \frac{d}{dr_\star} \right) \Psi_{\text{polar}}(r) , \tag{82}$$

$$K(r) = \left(\frac{l(l+1)}{2r} + \frac{d}{dr_\star} \right) \Psi_{\text{polar}}(r) . \tag{83}$$

Duality relation: It is a very lengthy computation to extract the electric and magnetic components of the Weyl tensor. The particular expressions in terms of the effective wave-functions are rather complicated and will not be included here. However, once this is done, it is a simple matter to verify that the two sectors are mutually related by electric/magnetic duality (see Bakas 2009a), i.e.,

$$E_{ab}^{\text{polar}} = B_{ab}^{\text{axial}} , \tag{84}$$

$$B_{ab}^{\text{polar}} = -E_{ab}^{\text{axial}} , \tag{85}$$

provided that the same boundary conditions are imposed on the axial and polar wave-functions so that they are the same (up to an arbitrary factor) and the frequencies ω are also the same. Taking into account the normalization above, the formulae match exactly provided that

$$\Psi_{\text{axial}} = \frac{i\omega}{2} \Psi_{\text{polar}}(r) . \tag{86}$$

Thus, axial/polar interchange provides explicit construction of the dual gravitons, in analogy with the construction of dual photons, i.e.,

$$\tilde{h}_{\mu\nu}^{\text{axial}} = h_{\mu\nu}^{\text{polar}} , \qquad \tilde{h}_{\mu\nu}^{\text{polar}} = h_{\mu\nu}^{\text{axial}} . \tag{87}$$

5. HOLOGRAPHIC IMPLICATIONS OF DUALITY

As application of these results, we describe the boundary manifestation of electric/magnetic duality for Maxwell theory and linearized gravity on AdS_4 space-time. Since all fields on the bulk are expressed in terms of the wave-function $\Psi(r)$, their values at the boundary follow by expanding $\Psi(r)$ as $r \to \infty$. Thus, we first briefly discuss the normalizable solution of the Schrödinger equation and its asymptotic expansion under general boundary conditions and then use the dictionary of AdS/CFT correspondence to obtain the boundary form of duality. We will end this section with a brief discussion of the energy-momentum/Cotton tensor duality for AdS_4 black-holes.

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5. 1. SOLVING THE SCHRÖDINGER EQUATION

The effective Schrödinger equation that governs the axial and polar sectors of both Maxwell theory and linearized gravity on AdS_4 space-time simplifies to

$$\left(-\frac{d^2}{dx^2} + \frac{l(l+1)}{\sin^2 x}\right)\Psi(x) = \Omega^2 \Psi(x) \tag{88}$$

by first expressing r in terms of $r_\star$, as given by

$$\tan\left(\sqrt{-\frac{\Lambda}{3}}\, r_\star\right) = \sqrt{-\frac{\Lambda}{3}}\, r\ , \tag{89}$$

and then letting, for convenience,

$$x = \sqrt{-\frac{\Lambda}{3}}\, r_\star\ , \qquad \Omega = \sqrt{-\frac{3}{\Lambda}}\, \omega\ . \tag{90}$$

Here, x assumes all values from 0 to $\pi/2$ as r varies from the origin $r = 0$ to spatial infinity $r = \infty$.

This problem can be transformed into a hypergeometric differential equation, so that the normalizable solution satisfying $\Psi(0) = 0$ at the origin $r = 0$ is provided by

$$\Psi(x) = \cos x\ \sin^{l+1} x\ F(a,\ b;\ c;\ \sin^2 x) \tag{91}$$

with coefficients

$$a = \frac{1}{2}\left(l + 2 + \Omega\right), \quad b = \frac{1}{2}\left(l + 2 - \Omega\right), \quad c = l + \frac{3}{2}\ . \tag{92}$$

Its behavior at spatial infinity is described by the following asymptotic expansion in powers of $1/r$,

$$\Psi(r) = I_0 + \frac{I_1}{r} + \frac{I_2}{r^2} + \frac{I_3}{r^3} + \frac{I_4}{r^4} + \cdots\ , \tag{93}$$

where the first two coefficient turn out to be

$$I_0 \;=\; \Gamma^{-1}\left(\frac{1}{2}(l+2+\Omega)\right)\Gamma^{-1}\left(\frac{1}{2}(l+2-\Omega)\right) \tag{94}$$

$$I_1 \;=\; -2\sqrt{-\frac{3}{\Lambda}}\,\Gamma^{-1}\left(\frac{1}{2}(l+1+\Omega)\right)\Gamma^{-1}\left(\frac{1}{2}(l+1-\Omega)\right) \tag{95}$$

up to an overall (irrelevant) numerical factor. The remaining coefficients are determined by I_0 and I_1.

The boundary conditions at $r = \infty$ are solely expressed in terms of I_0 and I_1. Since

$$I_0 = \Psi(r = \infty)\ , \qquad \frac{\Lambda}{3}I_1 = \frac{d\Psi}{dr_\star}(r = \infty)\ , \tag{96}$$

it follows that general boundary conditions (also called mixed or Robin) can be expressed in terms of the ratio

$$\frac{I_0}{I_1} = \gamma \tag{97}$$

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for fixed constant γ that can assume all values, including zero and infinity. Thus, the allowed spectrum of frequencies ω obeys a transcendental relation given by ratios of gamma functions and can only be solved numerical for general values of γ. In all cases, however, the frequencies come in pairs $(\omega, -\omega)$, as can be readily seen from the particular expressions of I_0 and I_1 in terms of products of gamma functions. Consequently, by appropriate superposition of them, real solutions can always be constructed.

In general, the boundary conditions for the axial and polar sectors of field theories on AdS_4 space-time can be independent from each other, and so is the spectrum of allowed frequencies, whereas the effective Schrödinger equations are the always the same. Therefore, the axial and polar problems will be isospectral if and only if both sectors satisfy the same boundary conditions.

5. 2. MAXWELL THEORY

We will be brief here, since the holographic manifestation of electric-magnetic duality for Maxwell theory is well studied and generally understood as acting on the space of three-dimensional (boundary) conformal field theories with $U(1)$ symmetry and a chosen coupling to a background gauge field (Witten 2003, but see also de Haro & Petkou 2008 and references therein).

In this case, it is more appropriate to define electric and magnetic fields with respect to the radial ADM decomposition,

$$\mathcal{E}_a = F_{ra} \ , \qquad \mathcal{B}_a = {}^\star F_{ra} \ , \tag{98}$$

and consider their boundary values as representing the one-point functions of a global symmetry current and the curl of the correspondince source, respectively. The two are interchanged by axial/polar duality that now acts naturally on the moduli space of boundary conformal field theories.

5. 3. LINEARIZED GRAVITY

We will compute the energy-momentum tensor using holographic renormalization (see, for instance, de Haro et al. 2001, Skenderis 2001, Skenderis 2002 and Balasub-ramanian & Kraus 1999). For this, we first consider the expression

$$\kappa^2 T_{ab}^{(r)} = K_{ab} - K \gamma_{ab} - 2\sqrt{-\frac{\Lambda}{3}} \gamma_{ab} + \sqrt{-\frac{3}{\Lambda}} \left(R_{ab}[\gamma] - \frac{1}{2} R[\gamma] \gamma_{ab} \right) \tag{99}$$

written in terms of the intrinsic and extrinsic curvature of the three-geometry γ_{ab} that appears at fixed r in the radial ADM decomposition of the four-dimensional metric. Here, $\kappa^2 = 8\pi G$ is the gravitational coupling. Since the metric acquires an infinite Weyl factor as r is taken to infinity, it is more appropriate to think of the AdS_4 boundary as a conformal class of boundaries and define the metric on it, $\mathcal{I}$, as

$$ds_{\mathcal{I}}^2 = \lim_{r \to \infty} \left(-\frac{3}{\Lambda r^2} \gamma_{ab} dx^a dx^b \right). \tag{100}$$

Then, the renormalized energy-momentum tensor on $\mathcal{I}$ is defined accordingly by

$$T_{ab} = \lim_{r \to \infty} \left(\sqrt{-\frac{\Lambda}{3}} \ r \ T_{ab}^{(r)} \right) \tag{101}$$

and it is finite, traceless and conserved.

When this is applied to linearized gravity in AdS_4 space-time, one obtains explicit expressions for T_{ab} under general boundary conditions. At the same time, the metric on $\mathcal{I}$, which also depends on the boundary conditions, is not conformally flat in general. Actually, one may characterize the deviation from the conformally flat case by computing the Cotton tensor of the boundary metric, defined as (see, for instance, Deser et al. 1982)

$$C^{ab} = \frac{1}{2\sqrt{-\det\gamma}} \left(\epsilon^{acd}\nabla_c R^b{}_d + \epsilon^{bcd}\nabla_c R^a{}_d \right) \; , \tag{102}$$

and obtain explicit expressions for C_{ab} under general boundary conditions. The details of the calculations will not be presented here, but will rather state the end result: using the electric and magnetic components of the Weyl tensor, which are now taken with respect to the radial ADM decomposition,

$$\mathcal{E}_{ab} = Z_{arbr} \; , \qquad \mathcal{B}_{ab} = {}^\star Z_{arbr} \; , \tag{103}$$

the following relation holds true for all boundary conditions,

$$\lim_{r\to\infty} \left(\frac{\Lambda}{3} \; r^3 \mathcal{E}_{ab} \right) = \kappa^2 T_{ab} \; , \tag{104}$$

$$\lim_{r\to\infty} \left(\frac{\Lambda^2}{9} \; r^3 \mathcal{B}_{ab} \right) = C_{ab} \tag{105}$$

Then, we have the following relation among the two distinct type of perturbations satisfying the same general boundary conditions for Ψ (see Bakas 2009a),

$$C_{ab}^{\text{axial}} = \kappa^2 T_{ab}^{\text{polar}} \; , \tag{106}$$

$$C_{ab}^{\text{polar}} = \kappa^2 T_{ab}^{\text{axial}} \; , \tag{107}$$

which follow from electric/magnetic duality on the bulk and give rise to the energy-momentum/Cotton tensor duality (also known as dual graviton correspondence in Compère & Marolf 2008 and de Haro 2009); see also Anderson 2001 for important earlier work on this subject.

5. 4.   EXTENSION TO ADS_4 BLACK-HOLES

We end this section with a brief summary of the situation in the presence of black holes in AdS_4 space-time having mass m and profile metric function

$$f(r) = 1 - \frac{2m}{r} - \frac{\Lambda}{3}r^2 \; . \tag{108}$$

Here, Newton's constant G is normalized to 1 for convenience. In this case there is no electric/magnetic duality among metric perturbations in the bulk, but there is still an energy-momentum/Cotton tensor duality at the boundary for appropriately chosen boundary conditions, as will be seen shortly. Of course, Maxwell equations on the AdS_4 Schwarzschild space-time exhibit electric/magnetic duality that simply exchanges axial and polar gauge fields, as for all spherically symmetric backgrounds.

The metric perturbations of AdS_4 Schwarzschild solution can still be separated into axial and polar components with opposite parity, as for AdS_4 space-time, based on spherical symmetry. However, the two sectors do not reduce to the same radial Schrödinger problem, but they rather give rise to two closely related equations (see Regge & Wheeler 1957, Zerilli 1970b, Chandrasekhar 1983, Cardoso & Lemos 2001, Moss & Norman 2002),

$$\left(-\frac{d^2}{dr_\star^2} + V_\pm(r)\right)\Psi_\pm(r) = \omega^2\Psi_\pm(r) \ , \tag{109}$$

as in supersymmetric quantum mechanics (see, for instance, Cooper et al. 1995). Here, $(-)$ refers to the axial sector and $(+)$ refers to the polar sector of the theory. More precisely, it turns out that

$$V_\pm(r) = W^2(r) \pm \frac{dW(r)}{dr_\star} + \omega_{\rm s}^2 \tag{110}$$

where the superpotential of the partner potentials is

$$W(r) = \frac{6mf(r)}{r[(l-1)(l+2)r + 6m]} + i\omega_{\rm s} \tag{111}$$

with

$$\omega_{\rm s} = -\frac{i}{12m}(l-1)l(l+1)(l+2) \ . \tag{112}$$

The axial and polar Schrödinger problems of black hole perturbations are often referred in the literature as Regge-Wheeler and Zerilli equations, respectively (see Regge & Wheeler 1957, Zerilli 1970b).

Supersymmetric partner potentials have the same energy spectrum (and, hence, the axial and polar perturbations of the metric have the same frequencies ω) provided that the corresponding wave-functions satisfy the following first order relations,

$$\left(\mp\frac{d}{dr_\star} + W(r)\right)\Psi_\pm(r) = i(\omega_{\rm s} \pm \omega)\Psi_\mp(r) \ . \tag{113}$$

Then, it is not appropriate to choose the same boundary conditions for $\Psi_\pm(r)$ at $r = \infty$, but rather impose supersymmetric partner boundary conditions that satisfy equation (113) at $r = \infty$. There is a privileged set of such boundary conditions that yield the static (conformally flat) boundary metric

$$ds_{\mathcal{I}}^2 = -dt^2 - \frac{3}{\Lambda}\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{114}$$

for one type of perturbations and a certain time-dependent (non-conformally flat) boundary metric for the perturbations with opposite parity. In either case one finds that the variation of the holographic energy-momentum tensor for black hole perturbations satisfying the boundary condition (114) is given by the Cotton tensor of the supersymmetric partner boundary metric (see Bakas 2009b), i.e.,

$$\kappa^2\delta T_{ab}^{\rm polar} = C_{ab}^{\rm axial} \ , \qquad \kappa^2\delta T_{ab}^{\rm axial} = C_{ab}^{\rm polar} \ . \tag{115}$$

Thus, although there is no gravitational duality for black hole perturbations, there is still a dual graviton correspondence at the boundary in certain cases. The result should be contrasted to the energy-momentum/Cotton tensor duality for perturbed AdS_4 space-time, which, as described above, is valid for all boundary conditions.

It is also interesting to note in this context that the special boundary conditions that account for the energy-momentum/Cotton tensor duality for AdS_4 black holes encompass the hydrodynamic modes of very large black holes that saturate the KSS bound on the ratio of shear viscosity to entropy density (see Kovtun et al. 2005),

$$\frac{\eta}{s} = \frac{1}{4\pi} \; . \tag{116}$$

Further details on these aspects can be found in the published work (see Bakas 2009b).

6. CONCLUSIONS AND DISCUSSION

We described the electric/magnetic duality of sourceless Maxwell equations and linearized Einstein equations in parallel. Using the theory of vector and tensor spherical harmonics, we have provided explicit construction of the dual photon and graviton configurations by simply exchanging axial and polar field configurations on spherically symmetric backgrounds. We also considered the holographic implications of duality for either Maxwell theory or linearized gravity on AdS_4 space-time in the context of AdS_4/CFT_3 correspondence. We expect to have similar duality relations for the two-point correlation functions of the energy-momentum tensor, as for the one-point functions, but not for higher correlators, since gravitational duality is only valid at the linear level. In any case, the boundary manifestation of gravitational duality as energy-momentum/Cotton tensor duality should be studied further and also extended to the coupled system of Einstein-Maxwell field equations. Generalizations of the duality rotations to higher spin fields were not discussed at all here, but clearly they can also be investigated using similar techniques based on generalized spherical harmonics.

There is a closely related result by Fefferman and Graham (see Fefferman & Graham 2007), which can be stated as energy-momentum/Cotton tensor *self-duality* in the context of the holographic manifestation of gravitational self-duality. In particular, self-dual gravitational (instanton) configurations in the bulk satisfying

$$^\star Z_{\mu\nu\rho\sigma}(g) = \pm Z_{\mu\nu\rho\sigma}(g) \tag{117}$$

have energy-momentum tensor equal to the Cotton tensor of their boundary metric γ_{ab},

$$\kappa^2 T_{ab}(\text{instantons}) = C_{ab}(\gamma) \; . \tag{118}$$

AdS_4 space-time satisfies trivially this relation, since both sides of the equation vanish identically, whereas perturbations around it are related by duality rotations that account for the energy-momentum/Cotton tensor duality at the boundary. Thus, it is conceivable, that small perturbations of gravitational instanton configurations will also exhibit duality relations (at least to linear order) and furthermore there will be a boundary relation of the form $\kappa^2 \delta T_{ab} = \delta C_{ab}$ among the perturbations of the holographic energy-momentum tensor and the Cotton tensor of the dual perturbed boundary metric. Work in this direction is in progress.

Finally, we end with some ideas about the plausible relevance of $\mathcal{H}$-space for our problems. Recall that the $\mathcal{H}$-space (also known as *heaven*) arose in the literature of general relativity by considering all shear-free cuts of the complexified future null infinity of an asymptotically flat Lorentzian four-manifold satisfying Einstein equations with $\Lambda = 0$ (see, for instance, Ko et al. 1981 and references therein). In this context, the original surface at infinity turns into the conformal infinity of a complex four-manifold with a holomorphic metric whose conformal curvature is self-dual. This construction was subsequently extended to space-times with a cosmological constant (see LeBrun 1982), with the appropriate technical ingredients. It may prove useful to reformulate the gravitational duality rotations in terms of complex analysis in $\mathcal{H}$-space and explore its holographic manifestation (when $\Lambda < 0$) using the self-duality of $\mathcal{H}$ in the spirit of Fefferman and Graham. The entire program of holographic renormalization on AdS_4 spaces may very well find a useful place in $\mathcal{H}$-space, turning heaven into powerful calculational tool. Work in this direction is also in progress.

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