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INTRODUCTION TO STRING THEORY

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Abstract. In these lectures, we will give a short introduction into some basics of string theory. The main focus will be on the bosonic string, as many ideas and methods can already be understood in this simple setting. However, we also review some aspects of the superstring and mention recent developments. More precisely, we discuss the quantization of the bosonic string and the relation to conformal field theory. We also touch upon more advanced topics such as D-branes, compactifications and T-duality. Finally, we give a broad overview of the superstring and mention some further topics of interest, relevant to current research.

1. MOTIVATION

Before getting into some of the details of string theory, let us start by a short motivation. There are different classes of questions or open problems that one hopes to understand better using string theory. These are

1. *Incomplete understanding of current theories*
   - (a) there are more than 20 free parameters in the Standard Model (SM) of particle physics (including the gauge couplings, theta angles, Yukawa couplings and the parameters in the Higgs potential)
   - (b) the flavor or family structure of the SM is poorly understood
   - (c) there are singularities in General Relativity (GR), most prominently in the description of black holes and the Big Bang
   - (d) quantization of gravity is ill understood (GR is non-renormalizable as a field theory)
2. *Unified description*
   - (a) string theory achieves a unified picture of nature by describing all "particles" as different excitations of one object, a string
   - (b) all consistent string theories contain both gravity and gauge theory; maybe this is necessary for consistency (this link between gravity and gauge theories is also at the heart of the AdS/CFT correspondence)

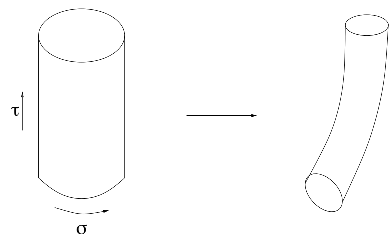

Figure 1: Embedding of the world-sheet, $\tau \in \mathbb{R}, \sigma \in [0, 2\pi]$.

3. *Phenomenology*; string theory might shed light on the nature of
   (a) dark matter
   (b) cosmic inflation
   (c) dark energy
4. *Other fundamental questions*
   (a) dimensionality of space-time
   (b) ...

In the following, we will give a short introduction into some basics of string theory. We start with a rather detailed description of the bosonic string, discussing it at the classical and the quantum mechanical level. Then we briefly generalize the methods to the superstring and end by some comments on recent developments.

2. THE BOSONIC STRING

We begin our discussion with the closed bosonic string. In order to describe its dynamics in a D-dimensional Minkowski space-time, we look for an action principle. One gets a hint for a starting point by analogy with the description of a point particle (of mass m). It moves along world-lines of minimal length, i.e. it minimizes the action

$$S = -m \int_{s_0}^{s_1} ds \ . \tag{1}$$

It is tempting to demand that a string move along world-surfaces with minimal area. To calculate the area, one needs a parametrization describing how the string is embedded into space-time $\mathcal{M}_D$, i.e. functions

$$X^\mu : \mathbb{R} \times S^1 \equiv \Sigma \longrightarrow \mathcal{M}_D \ , \tag{2}$$

cf. fig. 1 (we often use the world-sheet variables $\{\tau, \sigma\}$ and $\{\sigma^0, \sigma^1\}$ interchangeably).

The embedding induces a metric on the world-sheet, the pullback of the D-dimensional Minkowski metric,

$$h_{ab} = \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} \quad , \quad a, b \in \{0, 1\} \ , \ \mu, \nu \in \{0, \ldots, D-1\} \ . \tag{3}$$

As it arises via pullback of $\eta_{\mu\nu}$, it is not an independent variable. Demanding that the world-sheet have minimal area, amounts to minimizing the so called *Nambu-Goto action*

$$S_{\rm NG}[X] = -\frac{1}{2\pi\alpha'}\int d\sigma d\tau\sqrt{-h}\ , \tag{4}$$

where $h = \det(h_{ab})$ and α' has dimension (length)^2 (it is required to make the action dimensionless; $\sqrt{\alpha'}$ sets the string length). The Nambu-Goto (NG) action has a nice geometrical meaning, but it leads to non-linear equations of motion. Thus, one often takes the so called *Polyakov action* as the starting point which is classically equivalent. It treats the world-sheet metric as an additional independent degree of freedom and it is given by

$$S_{\rm Pol}[X,\gamma] = -\frac{1}{4\pi\alpha'}\int d\sigma d\tau\sqrt{-\gamma}\gamma^{ab}\partial_a X^\mu\partial_b X^\nu\eta_{\mu\nu}\ . \tag{5}$$

To see the equivalence with the NG-action consider the equations of motion for γ_{ab}: $0 = \frac{\delta S}{\delta\gamma^{ab}} = -\frac{\sqrt{-\gamma}}{4\pi\alpha'}T_{ab}$ with the energy momentum tensor

$$T_{ab} = \frac{1}{2}\partial_a X^\mu\partial_b X_\mu - \frac{1}{4}\gamma_{ab}\gamma^{cd}\partial_c X^\mu\partial_d X_\mu\ . \tag{6}$$

The equations of motion do not contain any derivatives of γ_{ab}. Rather, they imply $h_{ab} = \frac{1}{2}\gamma_{ab}\gamma^{cd}h_{cd}$ and $(-\gamma)^{1/2}\gamma^{ab} = (-h)^{1/2}h^{ab}$. Plugging this into $S_{\rm Pol}$, one obtains $S_{\rm NG}$.

Note that the energy momentum tensor is conserved and traceless, i.e.

$$\nabla^a T_{ab} = 0 \quad , \quad T_{ab}\gamma^{ab} = 0\ . \tag{7}$$

The action $S_{\rm Pol}$ enjoys several symmetries:

1. Local reparametrizations of the world-sheet

$$\sigma^a \to \tilde\sigma^a = \tilde\sigma^a(\sigma) \tag{8}$$

act as

$$\gamma_{ab}(\sigma) \to \tilde\gamma_{ab}(\tilde\sigma) = \frac{\partial\sigma^c}{\partial\tilde\sigma^a}\gamma_{cd}(\sigma)\frac{\partial\sigma^d}{\partial\tilde\sigma^b}\ , \tag{9}$$

while X^μ is unchanged. The presence of this symmetry is responsible for the conservation of T_{ab}.

2. Local Weyl-rescaling

$$\gamma_{ab}(\tau,\sigma) \to \gamma^w_{ab}(\tau,\sigma) = e^{w(\tau,\sigma)}\gamma_{ab}(\tau,\sigma). \tag{10}$$

The invariance of $S_{\rm Pol}$ under Weyl-rescaling is easily checked by noticing $\sqrt{-\gamma^w} = \sqrt{-\gamma}e^w$ and $(\gamma^w)^{ab} = \gamma^{ab}e^{-w}$. The invariance under Weyl-rescaling implies the tracelessness of T_{ab}.

Thus, there are three independent components in the world-sheet metric

$$\gamma_{ab} = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{12} & \gamma_{22} \end{pmatrix} \tag{11}$$

and three local gauge transformations. Therefore, locally one can choose the gauge

$$\gamma_{ab} = \eta_{ab} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{12}$$

In this gauge the action becomes that of free fields (hence the subscript "FF")

$$S_{\rm Pol} \to S_{\rm FF}[X] = -\frac{1}{4\pi\alpha'} \int d\sigma d\tau \eta^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} . \tag{13}$$

This leads to the equations of motion

$$\partial_\tau^2 X^\mu - \partial_\sigma^2 X^\mu = 0 \tag{14}$$

plus the constraints (from the equations of motion of γ_{ab})

$$T_{ab} = \frac{1}{2}\partial_a X^\mu \partial_b X_\mu - \frac{1}{4}\eta_{ab}\partial_c X^\mu \partial^c X_\mu = 0 . \tag{15}$$

Note that choosing the gauge (12) does not fix all the gauge freedom yet. This can be easily seen by introducing light cone coordinates

$$\sigma^\pm = \tau \pm \sigma , \tag{16}$$

in which the metric takes the form $ds^2 = -d\sigma^+ d\sigma^-$. Any transformation of the form

$$\tilde\sigma^+ = f_+(\sigma^+) \quad , \quad \tilde\sigma^- = f_-(\sigma^-) \tag{17}$$

would leave the metric invariant up to a Weyl-transformation, i.e.

$$ds^2 = -\frac{1}{f'_+ f'_-} d\tilde\sigma^+ d\tilde\sigma^- \stackrel{\rm Weyl}{\longrightarrow} -d\tilde\sigma^+ d\tilde\sigma^- . \tag{18}$$

A combination of a diffeomorphism of the form (17) together with a Weyl-transformation that leaves the metric (12) invariant, is called a *conformal transformation* and the gauge choice (12) is accordingly called the *conformal gauge*.

The general solution to (14) (before imposing the constraints), is easiest to find using light-cone coordinates on the world-sheet. The equations of motion become

$$\partial_+ \partial_- X^\mu = 0 . \tag{19}$$

They have the general solution

$$X^\mu(\tau,\sigma) = X_1^\mu(\sigma^+) + X_2^\mu(\sigma^-) , \tag{20}$$

or, in Fourier space,

$$X^\mu(\tau,\sigma) = x_0^\mu + \underbrace{\sqrt{\frac{\alpha'}{2}}(\tilde\alpha_0^\mu \sigma^+ + \alpha_0^\mu \sigma^-)}_{\alpha' p^\mu \tau} + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}(\tilde\alpha_n^\mu e^{-in\sigma^+} + \alpha_n^\mu e^{-in\sigma^-}) . \tag{21}$$

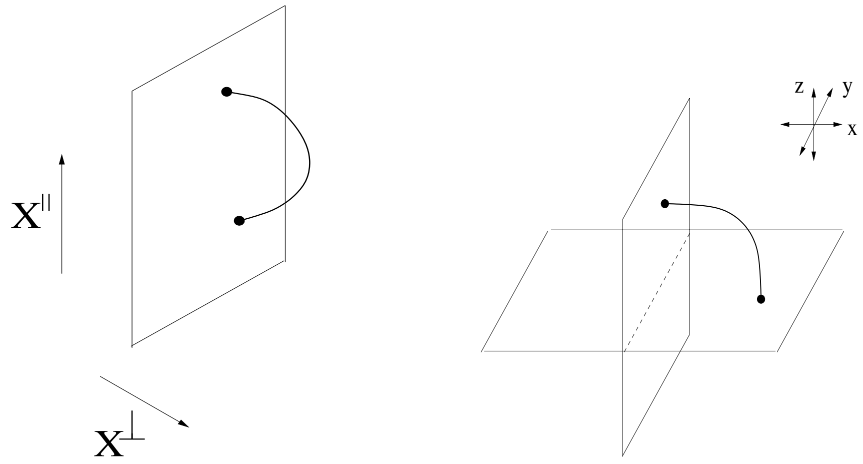

Figure 2: Different boundary conditions of open strings. In the left figure, the string has NN-boundary conditions along $X^{||}$ and DD-boundary conditions along $X^{\perp}$. In the right figure, it is NN along y, ND along z and DN along x.

The introduction of the center of mass momentum p^μ is possible because one has to demand periodicity (under $\sigma \to \sigma + 2\pi$, i.e. $\sigma^\pm \to \sigma \pm 2\pi$)

$$X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma+2\pi) \quad \Longrightarrow \quad \tilde{\alpha}_0^\mu = \alpha_0^\mu \equiv p^\mu \sqrt{\frac{\alpha'}{2}} \ . \tag{22}$$

Thus, the general solution (of the unconstraint equations of motion) is a sum of left and right moving waves on the string, which are only coupled via the center of mass momentum.

Demanding reality of the coordinate functions X^μ results in the relations

$$(\alpha_n^\mu)^* = \alpha_{-n}^\mu \quad , \quad (\tilde{\alpha}_n^\mu)^* = \tilde{\alpha}_{-n}^\mu \quad , \quad (p^\mu)^* = p^\mu \quad , \quad (x^\mu)^* = x^\mu \ . \tag{23}$$

Before discussing the constraints that one still has to impose on the general solution, let us first have a quick look at the open string. Its action is the same as for the closed string, but σ now takes values in the interval $\sigma \in [0,\pi]$ instead of a circle. Thus, varying the action now leads to a boundary term

$$\delta S \sim \int_{\delta\Sigma} d\tau (\partial_\sigma X^\mu) \delta X_\mu \ . \tag{24}$$

There are four different possibilities to cancel this,

$$\left\{ \begin{array}{c} \partial_\sigma X^\mu = 0 \\ \delta X^\mu = 0 \end{array} \right\} \begin{array}{c} \text{(Neumann)} \\ \text{(Dirichlet)} \end{array} \ \text{at} \ \ \sigma = \left\{ \begin{array}{c} 0 \\ \pi \end{array} \right\} \ . \tag{25}$$

Neumann (N) boundary conditions at the end of the string imply that the endpoint can move freely in that direction and momentum is conserved during the movement of that end of the string, whereas Dirichlet (D) conditions imply that the string end point is fixed in that direction. If there are p spatial dimensions in which the string can move freely, one says that the string is ending on an object called a *Dirichlet p-brane*, or *Dp-brane* for short. This is depicted in figure 2.

The general solution for the open string can be found analogously to the closed string, imposing in addition the different boundary conditions (25). For instance, in the case of NN boundary conditions one would find

$$X^\mu(\tau,\sigma) = x_0^\mu + (2\alpha')p^\mu\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}e^{-in\tau}\cos(n\sigma) \ . \tag{26}$$

On the general solutions of the closed and open string, (21) and (26), one still has to impose the constraints $T_{ab} = 0$. This is simplified by noticing the following features of the energy momentum tensor:

1. T_{ab} traceless: $0 = T_{ab}\gamma^{ab} \sim T_{-+} + T_{+-} = 2T_{+-} = 2T_{-+}$.
2. T_{ab} conserved: $\partial_- T_{++} = 0 = \partial_+ T_{--} \Rightarrow T_{\pm\pm} = T_{\pm\pm}(\sigma^\pm) = \frac{1}{2}\partial_\pm X^\mu \partial_\pm X_\mu$.

Thus, one has to impose $T_{++}(\sigma^+) = 0 = T_{--}(\sigma^-)$ on the solutions.

Often one uses Fourier modes

$$\begin{aligned} L_m &= \frac{1}{2\pi\alpha'}\int_0^{2\pi} d\sigma e^{-im\sigma} T_{--} = \frac{1}{2}\sum_n \alpha_{m-n}\cdot\alpha_n \ , \\ \tilde{L}_m &= \frac{1}{2\pi\alpha'}\int_0^{2\pi} d\sigma e^{im\sigma} T_{++} = \frac{1}{2}\sum_n \tilde{\alpha}_{m-n}\cdot\tilde{\alpha}_n \end{aligned} \tag{27}$$

with which one can write the components of the energy momentum tensor as

$$T_{--} = \sum_{m\in\mathbb{Z}} L_m e^{-im\sigma^-} \quad , \quad T_{++} = \sum_{m\in\mathbb{Z}} \tilde{L}_m e^{im\sigma^+} \ . \tag{28}$$

For the open string one only has one set of Fourier modes, denoted by L_m. Using Fourier modes (which are also known as *Virasoro modes* in the present context), the constraints become

$$\forall_{m\in\mathbb{Z}} : L_m = 0 = \tilde{L}_m \ . \tag{29}$$

This leads to subtleties when quantizing as we will see in a moment.

2. 1. QUANTIZATION

String theory is usually quantized using first quantization. One imposes canonical commutation relations on X^μ and the canonical momenta

$$\Pi^\mu = \frac{1}{2\pi\alpha'}\partial_\tau X^\mu \left(= \frac{\partial\mathcal{L}}{\partial(\partial_\tau X^\mu)}\right) \ . \tag{30}$$

More precisely,

$$[X^\mu(\tau,\sigma), \Pi^\nu(\tau,\sigma')] = i\eta^{\mu\nu}\delta(\sigma-\sigma') \ . \tag{31}$$

Plugging in the general solution (21), one finds

$$[x^\mu, p^\nu] = i\eta^{\mu\nu} \quad , \quad [\alpha_m^\mu, \alpha_n^\nu] = m\eta^{\mu\nu}\delta_{m+n,0} = [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] \quad , \quad [\alpha_m^\mu, \tilde{\alpha}_n^\nu] = 0 \ . \tag{32}$$

Hermiticity implies

$$(\alpha_m^\mu)^\dagger = \alpha_{-m}^\mu \ . \tag{33}$$

Thus, the commutation relations (32) are equivalent (for each $m > 0$) to the algebra of creation and annihilation operators of a harmonic oscillator. This can be made even more apparent by redefining

$$\forall_{m>0} : \left[a_m^\mu \equiv \frac{1}{\sqrt{m}} \alpha_m^\mu \quad , \quad a_m^{\dagger\mu} \equiv \frac{1}{\sqrt{m}} \alpha_{-m}^\mu \right] \ , \tag{34}$$

which implies

$$[a_m^\mu, a_n^{\dagger\nu}] = \eta^{\mu\nu}\delta_{m,n} \ . \tag{35}$$

Using these operators, one can build a Fock space, where α_m^μ is a lowering operator for $m > 0$ and a raising operator for $m < 0$. The oscillator ground states which are eigenstates of the momentum operator are defined as

$$\forall_{m>0} : \left[\alpha_m^\mu |0; p^\nu\rangle = 0 \quad \text{and} \quad \tilde{\alpha}_m^\mu |0; p^\nu\rangle = 0 \right] \ . \tag{36}$$

Using these ground states one obtains a basis of the full Fock space by acting with the raising operators, i.e.

$$\left\{ \alpha_{-m_1}^{\mu_1} \ldots \tilde{\alpha}_{-n_1}^{\nu_1} \ldots |0; p^\mu\rangle \middle| m_i > 0, n_i > 0 \right\} \ . \tag{37}$$

However,

$$\forall_{m>0} : \langle 0|\alpha_m^0 \alpha_{-m}^0|0\rangle = \langle 0|[\alpha_m^0, \alpha_{-m}^0]|0\rangle = -m < 0 \ , \tag{38}$$

i.e. there are negative norm states in the Fock space we just constructed. The solution is that one still has to impose the constraints. There are two equivalent ways to do so. One can either impose the constraints before or after quantization. In the first case one is led to the method of *light cone quantization*, the second procedure is known as *covariant quantization* (i.e. one imposes the constraints on the Fock space, which is similar to the Gupta-Bleuler method of QED). Here we will discuss only the light cone quantization as it allows to obtain the spectrum of the string in the most straightforward way.

In this approach, before quantization one imposes a condition on X^μ in order to fix the remaining gauge freedom (cf. (17))

$$\tilde{\tau} = \frac{1}{2}\left[\tilde{\sigma}^+(\sigma^+) + \tilde{\sigma}^-(\sigma^-)\right] \quad , \quad \tilde{\sigma} = \frac{1}{2}\left[\tilde{\sigma}^+(\sigma^+) - \tilde{\sigma}^-(\sigma^-)\right] \ . \tag{39}$$

Obviously, $\tilde{\tau}$ obeys the same equation as the X^μ's

$$\partial_+\partial_-\tilde{\tau} = 0 \ . \tag{40}$$

This allows to identify $\tilde{\tau}$ with one of them. This is most conveniently done using light cone coordinates in space-time, i.e.

$$X^{\pm} = \frac{1}{\sqrt{2}}\left(X^0 \pm X^{D-1}\right) . \tag{41}$$

In these coordinates, the Minkowski metric takes the form

$$\eta_{\mu\nu} = \begin{pmatrix} 0 & -1 & & & \\ -1 & 0 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} . \tag{42}$$

Now the light cone gauge is

$$X^+(\tau, \sigma) = \alpha' p^+ \tau . \tag{43}$$

This fixes also $\tilde{\sigma}$ (up to rigid translations for the closed string). This choice of gauge allows to solve the constraints explicitly. For a flat world sheet metric the constraints (15) are

$$(\dot{X} \pm X')^2 = 0 , \tag{44}$$

where the dot stands for derivatives w.r.t. τ and the prime for derivatives w.r.t. σ. In (space-time) light cone coordinates (using the metric (42)), these become

$$0 = -2 \underbrace{\left(\dot{X}^+ \pm X^{+\prime}\right)}_{\alpha' p^+} \left(\dot{X}^- \pm X^{-\prime}\right) + \left(\dot{X}^I \pm X^{I\prime}\right)^2 . \tag{45}$$

As this is linear in X^-, one can solve for

$$\dot{X}^- \pm X^{-\prime} = \frac{1}{2p^+\alpha'}\left(\dot{X}^I \pm X^{I\prime}\right)^2 , \tag{46}$$

or, in terms of the Fourier modes

$$\alpha_n^- = \frac{1}{\sqrt{2\alpha'}p^+} \sum_{m=-\infty}^{\infty} \alpha_{n-m}^I \alpha_m^I = \frac{2}{\sqrt{2\alpha'}p^+} L_n^{\perp} , \tag{47}$$

where we defined the transversal Virasoro mode $L^{\perp}$, and similarly for $\tilde{\alpha}_n^-$. Note that for this simplification it was crucial, both to fix the gauge by identifying τ with one of the coordinate fields, cf. (43) (which simplified the form of $\dot{X}^+ \pm X^{+\prime}$ in (45)) and to use light cone coordinates in space-time, which rendered the constraint (45) linear in $\dot{X}^- \pm X^{-\prime}$.

¿From (46) it is apparent that the independent degrees of freedom are

$$X^I, \dot{X}^I, x_0^-, p^+ \ (= p^0 + p^1) , \tag{48}$$

where x_0^- is the zero-mode that is not determined by the constraints.

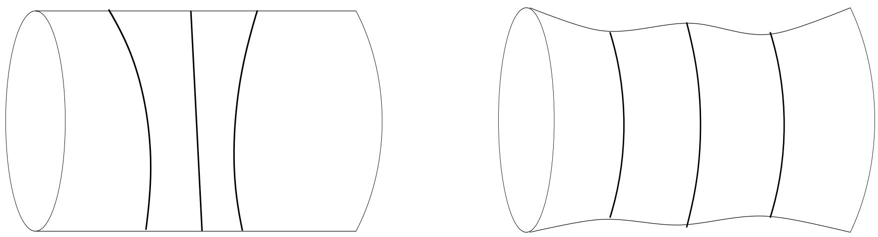

Figure 3: Longitudinal and transversal oscillations

This has a natural interpretation. The worldsheet reparametrization invariance removes all degrees of freedom along the lightcone and can be used to absorb oscillations tangential to the string, leaving only the transversal oscillations as physical, cf. fig. 3. This is to be contrasted with the non-relativistic classical string.

Having fixed the residual gauge in this fashion, we can now proceed as above and impose canonical commutation relations only for the independent operators (48). Now the oscillator ground states are given by $|0; p^+, p^I\rangle$ and the excited states are given by

$$\left\{ \alpha^{I_1}_{-m_1} \dots \tilde{\alpha}^{J_1}_{-n_1} \dots |0; p^+, p^I\rangle \middle| m_i > 0, n_i > 0 \right\} \ . \tag{49}$$

A few comments are in order. The Fock space in light cone gauge (49) does not contain any negative norm states anymore. Moreover, for the closed string there is one constraint left that still has to be imposed, i.e.

$$\alpha^-_0 = \tilde{\alpha}^-_0 \ . \tag{50}$$

Ultimately, this arises due to the fact that we did not fix all the gauge freedom. As mention after eq. (43), $\tilde{\sigma}$ is not completely fixed yet: the origin along the string worldsheet can still be chosen freely.

Canonical quantization elevates the transversal Virasoro modes to operators and, thus, one needs a prescription for the ordering in their definition. The definition

$$L^\perp_n = \frac{1}{2} \sum_{m=-\infty}^{\infty} \alpha^I_{n-m} \alpha^I_m \quad , \quad \tilde{L}^\perp_n = \frac{1}{2} \sum_{m=-\infty}^{\infty} \tilde{\alpha}^I_{n-m} \tilde{\alpha}^I_m \tag{51}$$

is only unambiguous for $n \neq 0$, whereas for $n = 0$ an ambiguity arises due to (32). One just defines the *normal ordered* form

$$L^\perp_0 \equiv \underbrace{\frac{1}{2} \alpha^I_0 \alpha^I_0}_{\frac{\alpha'}{4} p^I p^I} + \underbrace{\sum_{m=1}^{\infty} \alpha^I_{-m} \alpha^I_m}_{\equiv N^\perp} \ , \tag{52}$$

and analogously for $\tilde{L}^\perp_0$, but takes into account the ordering ambiguity by generalizing the relation (47), allowing a normal ordering constant a, i.e.

$$\overset{(\sim)}{\alpha}{}^{-}_0 = \frac{2}{\sqrt{2\alpha'} p^+} \left(\overset{(\sim)}{L}{}^{\perp}_0 + a \right) \ . \tag{53}$$

In (52), we defined the (transversal) number operator $N^\perp$ which counts the number of transversal excitations, $N^\perp \alpha^I_{-m}|0\rangle = m\alpha^I_{-m}|0\rangle$ (and similar for $\tilde{N}^\perp$). In (53) the normal ordering constant a is the same for $L_0^\perp$ and $\tilde{L}_0^\perp$, as both of them have the same ambiguity. Using the remaining constraint (50), this translates to

$$N^\perp = \tilde{N}^\perp \ , \tag{54}$$

i.e. the numbers of (transversal) excitations of the right- and the left-movers have to coincide.

The normal ordering constant a in (47) has a crucial effect on the mass spectrum, which is given by

$$M^2 = -p^I p^I + 2p^+p^- = \left\{ \begin{array}{cc} \frac{1}{\alpha'}(N^\perp + a) & \text{(open)} \\ \frac{2}{\alpha'}(N^\perp + \tilde{N}^\perp + 2a) & \text{(closed)} \end{array} \right. \ , \tag{55}$$

where we used (52), (53) and the fact that $p^- = \sqrt{\frac{2}{\alpha'}}\alpha_0^- = \sqrt{\frac{2}{\alpha'}}\tilde{\alpha}_0^-$. Let us have a closer look at the consequences of the appearance of the normal ordering constant in the mass formula (55), starting with the

- *Open string*

  1. The ground state $|0, p^+, p^I\rangle \equiv |0\rangle$ has mass $\alpha' M^2 = a$.
  2. 1. excited level: The states $\alpha^I_{-1}|0\rangle$ form a $(D-2)$-dimensional vector of $SO(D-2)$. Physical states fall into representations of the little group of the Lorentz group $SO(D-1,1)$. The little group is the subgroup of $SO(D-1,1)$ which leaves the momentum vector of the state under consideration invariant. The momentum vector of a massive state ($-p^2 = M^2$) in its rest frame is given by $p^\mu \sim (M, 0, \ldots, 0)$ and, thus, the little group is $SO(D-1)$. For a massless state ($-p^2 = 0$), the momentum vector can instead be chosen as $p^\mu \sim (M, M, 0, \ldots, 0)$, leading to the little group $SO(D-2)$. Obviously, the states at the first excited level have to be massless then, as they form the vector representation of $SO(D-2)$. This allows us to determine the normal ordering constant a, as the mass formula (55) implies

  $$\alpha' M^2 \alpha^I_{-1}|0\rangle = (1+a)\alpha^I_{-1}|0\rangle \ . \tag{56}$$

  Setting this to zero, leads to

  $$a = -1 \ , \tag{57}$$

  showing that the ground state is actually tachyonic (i.e. it has a negative mass squared). This problem of the bosonic string is overcome in the superstring, as we will review in a moment.
  3. 2. excited level: At the second excited level there are two types of states, $\alpha^I_{-2}|0\rangle$ and $\alpha^I_{-1}\alpha^J_{-1}|0\rangle$. The first set of states transforms as a vector and the second as a symmetric tensor of $SO(D-2)$. However, they combine to form a symmetric traceless tensor of $SO(D-1)$, which is the appropriate little group for massive states.

- *Closed string*
  1. The ground state is again a tachyon, this time of mass $\alpha' M^2 = -4$.
  2. 1. excited level: Due to the constraint (54) the first excited states are $\alpha^I_{-1}\tilde{\alpha}^J_{-1}|0\rangle$, with mass $\alpha' M^2 = 4(1+a) = 0$. Thus, these states should fall into representations of the little group for massless states, i.e. $SO(D-2)$. Indeed, they decompose according to

$$\begin{aligned}\alpha^I_{-1}\tilde{\alpha}^J_{-1}|0\rangle &= \alpha^{[I}_{-1}\tilde{\alpha}^{J]}_{-1}|0\rangle + \left[\alpha^{(I}_{-1}\tilde{\alpha}^{J)}_{-1}|0\rangle - \frac{1}{D-2}\delta^{IJ}\alpha^K_{-1}\tilde{\alpha}^K_{-1}\right]|0\rangle \\ &\quad + \frac{1}{D-2}\delta^{IJ}\alpha^K_{-1}\tilde{\alpha}^K_{-1}|0\rangle \end{aligned} \tag{58}$$

  into an antisymmetric tensor, a symmetric traceless tensor and a scalar. In covariant language, these correspond to the so called *Kalb-Ramond* tensor $B_{\mu\nu}$, the graviton $g_{\mu\nu}$ and the dilaton ϕ.

Several comments are in order here. The massless spectrum contains a graviton (from the closed string) and a vector (from the open string) and, thus, the theory is able to describe gravity and Yang-Mills theory (of course one still has to check that the fields are also evolving according to the appropriate dynamics; a more sophisticated analysis shows that they indeed do). This remarkable fact, however, crucially depends on having a normal ordering constant a equal to -1. Without a normal ordering ambiguity (i.e. in a classical theory of strings) there would not be any shift in the mass formula (55) and all massless fields would be scalars. Thus, the description of phenomenologically interesting theories like gravity and Yang-Mills theory requires the use of quantized strings as opposed to classical strings.

Even though the presence of the graviton and a vector as excitations of the string is promising, there are two obvious problems with the spectrum. The lowest mass eigenstate is tachyonic (indicating that Minkowski space in the bosonic theory is unstable) and there are no massless fermions. The absence of fermions continues to hold at higher excitation numbers and in any case, in order to describe the fermions of the Standard Model one would like to find massless fermions which obtain their masses via the Higgs mechanism. The masses of the excited string states are quantized in inverse powers of the string length, and, thus, too high for Standard Model particles.

A last remark concerns the dimensionality of space-time. Until now, we kept D as an undetermined parameter. It turns out that this parameter is actually fixed by demanding consistency of the theory. One can show that in light cone quantization, while unitarity is manifest, Lorentz-invariance is broken unless $D = 26$. This number changes to $D = 10$ for the superstring.

2. 2. INTERACTIONS

In order to describe the dynamics of the (massless) modes, one has to take into account interactions and calculate scattering amplitudes. This can be done using the (Euclidean) path integral formalism. Certainly, this goes beyond the scope of these lectures. Let us just make a few comments here.

At weak string coupling, the scattering amplitudes can be expanded in a perturbation series, similar to Quantum Field Theory (QFT). For the closed string the first

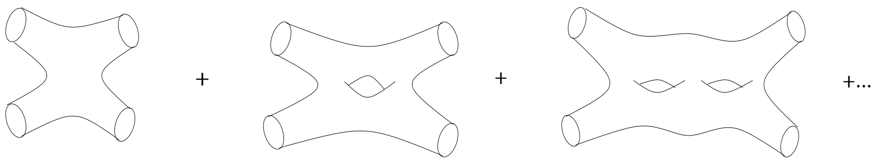

Figure 4: Closed string perturbation series.

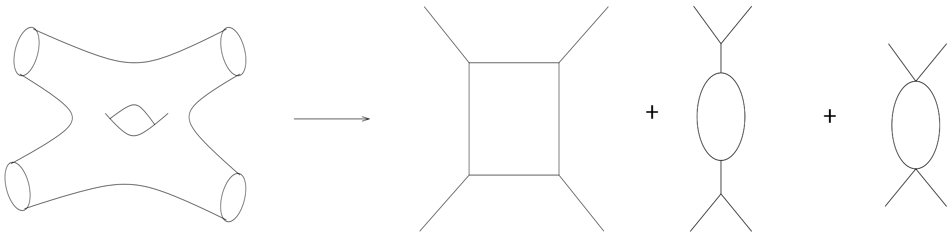

Figure 5: The field theory limit.

few contributions in this series are given in figure 4. When taking the limit of infinite string tension (or zero string length), the string interaction diagrams reduce to ordinary Feynman diagrams. A single string diagram can give rise to different field theory Feynman diagrams, as depicted in figure 5. This is due to the fact that the string diagram can be deformed in many different ways, because there are no definite interaction points of the strings. As the infinities of QFT arise when interaction points get close to each other (or in momentum space, from high energies), the absence of definite interaction points in string theory provides an intuitive understanding of why strings avoid the usual divergencies of QFTs.

The intersection points of QFT have another consequence. For example, there can be 3- or 4-point vertices as in figure 5 and at each vertex one can prescribe a different coupling strength and specify different species of particles that interact with each other. This leads to a large non-uniqueness of QFTs. In string theory this ambiguity does not exist, as there is basically only one kind of vertex (for closed strings), a 3-string vertex describing the splitting of a string into two or the joining of two strings to one. This leads to the high uniqueness of string theory, at least when describing strings moving in Minkowski space. As we mentioned before, quantized strings have to move in either 26 or 10 dimensions (for the bosonic and supersymmetric string, respectively) and the non-uniqueness enters string theory only when trying to make contact to the four-dimensional physics. This requires to consider all but four of the dimensions to be very small, so small that they were not observed yet. The choice of the form of these tiny dimensions in this process of *compactification* introduces a high degree of non-uniqueness of the four-dimensional effective theory. This is know as the *vacuum problem* of string theory. We will come back to string compactifications shortly.

Analyzing the scattering amplitudes of string theory allows to determine the interactions between the different massless fields. Without going into the details, here we

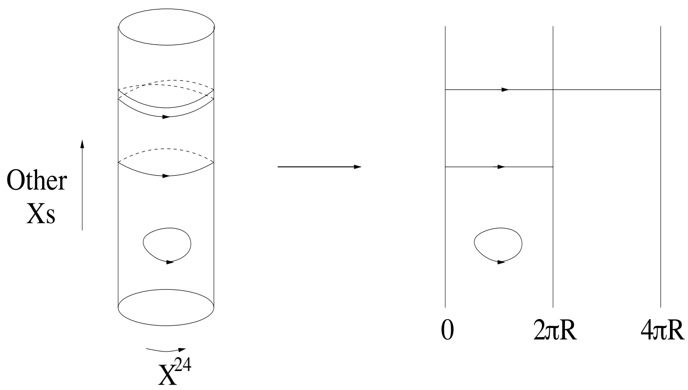

Figure 6: Strings with different winding numbers on a circle.

just display the resulting 26-dimensional action that one finds for the closed bosonic string, governing the dynamics of the Kalb-Ramond tensor, the graviton and the dilaton that we found in the last section to be the massless degrees of freedom. It is given by

$$S = \frac{1}{2\kappa^2} \int d^{26}x \sqrt{-g} e^{-2\phi} \left[R - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + 4\partial_\mu \phi \partial^\mu \phi + \mathcal{O}(\alpha') \right] , \tag{59}$$

where $H_{\mu\nu\rho} \sim \partial_{[\mu} B_{\nu\rho]}$ is the field strength of the antisymmetric tensor and $\mathcal{O}(\alpha')$ stands for higher derivative corrections of the form R^2, $R_{\mu\nu}R^{\mu\nu}$, $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, $(\partial_\mu\phi\partial^\mu\phi)^2$, $RH_{\mu\nu\rho}H^{\mu\nu\rho}, \ldots$. These correction terms are often negligibly small at low energies. It is important to note, however, that string theory predicts a well prescribed set of higher derivative corrections which are suppressed at low energies by factors of $\alpha' p^2$, where p^2 is the typical momentum or energy scale of the process under scrutiny. For the superstring one obtains similar actions, describing supergravity at low energies. Often these space-time actions are the starting point for phenomenological analyses in string theory.

2. 3. COMPACTIFICATION ON S^1

Before passing over to the superstring, we would like to take up again the issue of compactification. To exemplify the idea we only consider the toy example of a single compactified direction, which would still leave 25 large dimensions in the case of the bosonic string. To be concrete, we discuss the case in which the direction X^{24} (which we denote simply by X) takes the form of a circle of radius R, i.e.

$$X^{24} \equiv X = X + 2\pi R . \tag{60}$$

In contrast to the case, where the string moved in uncompactified Minkowski space, it can now wind around the circle direction an arbitrary number of times. Examples of singly and doubly wound strings are shown in figure 6.

Taking into account the winding number of the string ($\omega = mR\alpha'^{-1}$ with $m \in \mathbb{Z}$)

the general solution for the compact coordinate field is

$$\begin{aligned} X &= x_0 + \alpha' p\tau + \alpha' \omega\sigma + \text{oscillators} \qquad (61)\\ &= x_0 + \frac{\alpha'}{2}\underbrace{(p+\omega)}_{\sqrt{\frac{2}{\alpha'}}\tilde{\alpha}_0}\sigma^+ + \frac{\alpha'}{2}\underbrace{(p-\omega)}_{\sqrt{\frac{2}{\alpha'}}\alpha_0}\sigma^- + \text{oscillators}\ . \end{aligned}$$

In contrast to before, in the presence of non-trivial winding, α_0 and $\tilde{\alpha}_0$ are not identical and proportional to the center of mass momentum p anymore.

As the wavefunction of a string mode moving with momentum p along the compactified direction is proportional to $e^{2\pi ipx}$ and as this wavefunction ought to be single valued, the momentum along the circle direction is quantized, $p = \frac{n}{R}$.

Splitting the transversal coordinates into $I = \{i, 24\}$, the transversal Virasoro mode operator $L_0^\perp$ becomes

$$L_0^\perp = \frac{1}{2}\alpha_0^I\alpha_0^I + N^\perp = \frac{\alpha'}{4}p^ip^i + \frac{1}{2}\alpha_0\alpha_0 + N^\perp\ , \qquad (62)$$

and similarly for $\tilde{L}_0^\perp$. We first notice that in the presence of non-trivial winding, the constraint (54) on the right- and left-moving excitations does not hold anymore. This can easily be seen from (50), which now implies $0 = L_0^\perp - \tilde{L}_0^\perp = -\alpha' p\omega + N^\perp - \tilde{N}^\perp$.

Let us finally look at the mass spectrum for a string moving in a space-time of the form $\mathbb{R}^{1,24} \times S^1$, as this will reveal a truly remarkable feature of string theory. Using (62) and the fact that $p^- = \frac{1}{\alpha' p^+}(L_0^\perp + \tilde{L}_0^\perp - 2)$, the 25-dimensional mass squared becomes

$$\begin{aligned} M^2 = 2p^+p^- - p^ip^i &= \frac{1}{\alpha'}(\alpha_0\alpha_0 + \tilde{\alpha}_0\tilde{\alpha}_0) + \frac{2}{\alpha'}(N^\perp + \tilde{N}^\perp - 2)\\ &= \left(\frac{n}{R}\right)^2 + \left(\frac{mR}{\alpha'}\right)^2 + \frac{2}{\alpha'}(N^\perp + \tilde{N}^\perp - 2)\ . \qquad (63) \end{aligned}$$

It is easy to see that this mass spectrum is invariant under the transformation

$$R \longleftrightarrow \frac{\alpha'}{R}\ , \qquad (64)$$

which exchanges a small radius with a large one. For small radius the states with momentum along the compactified direction are very heavy and the winding states are very light. For large radius it is the other way round. The winding strings obtain a large mass due to the tension of the string and the momentum states become light. This invariance of the spectrum under (64) can be shown to extend to the interactions of the theory and, thus, it is an invariance of the full theory. This kind of immunity of the theory under changing some feature of space-time, i.e. the target space of the string embedding, is known as target space duality or *T-duality* for short.

We stress that the presence of the winding strings is crucial for T-duality to work. There is no analogous effect in any field theory. T-duality is the simplest example that shows that strings see space-time very differently from point particles. In the superstring there are many generalizations of this fascinating fact. The so called *mirror symmetry* even asserts that strings moving in space-times with different topologies can lead to the same effective physics (in the example at hand, the radius of the circle is changed, but the topology of space-time is the same for both radii).

3. SUPERSTRINGS

In this section we give a very short and sketchy introduction into superstring theory. For many of the details we have to refer to the references at the end of this article.

As mentioned before, two drawbacks of the bosonic string are that its spectrum contains a tachyon and no fermions. Both of these unwanted features are absent in supersymmetric versions of string theory. There are two different ways to introduce supersymmetry, either on the world-sheet or in the target space-time. In the first case, which leads to the *Ramond-Neveu-Schwarz* (RNS) superstring, one extends the Polyakov action (5) by adding world-sheet fermions to the space-time coordinate fields. In the second case, known as the *Green-Schwarz* superstring, the additional degrees of freedom on the world-sheet are scalars from the point of view of the world-sheet theory, but fermions from the point of view of space-time. Here we will only discuss the RNS superstring, in which supersymmetry in space-time is not manifest (but it arises indirectly).

In the RNS formalism, the world-sheet metric and the coordinate fields each have superpartners,

$$(\gamma_{ab}, X^\mu) \to (\gamma_{ab}, \chi_{a\alpha}, X^\mu, \psi^\mu_\alpha) \ , \tag{65}$$

where $\chi_{a\alpha}$ is the gravitino, the gauge field of local supersymmetry. The treatment of the RNS superstring closely mimics the procedure employed in the bosonic string. The local (super)symmetries can be used to choose the so called *superconformal gauge*

$$\gamma_{ab} = \eta_{ab} \ , \chi_{a\alpha} = 0 \ . \tag{66}$$

In this gauge, the world-sheet action takes the form

$$S_{RNS} = -\frac{1}{4\pi\alpha'} \int d\sigma d\tau \Big(\partial_a X^\mu \partial^a X_\mu + 2i\bar{\psi}^\mu \rho^a \partial_a \psi_\mu \Big) \ , \tag{67}$$

where ρ^a are the 2-dimensional Dirac-matrices

$$\rho^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad , \quad \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ . \tag{68}$$

Varying the action, i.e. $\delta S = 0$, leads to the equations of motion and some boundary conditions. The equations of motion in light cone coordinates are

$$\partial_+ \partial_- X^\mu = 0 \quad , \quad \partial_+ \psi^\mu_- = 0 = \partial_- \psi^\mu_+ \ , \tag{69}$$

where

$$\psi^\mu = \begin{pmatrix} \psi^\mu_+ \\ \psi^\mu_- \end{pmatrix} \ . \tag{70}$$

It is obvious that the general solution to (69) is given by

$$\psi^\mu_+ = \psi^\mu_+(\sigma_+) \quad , \quad \psi^\mu_- = \psi^\mu_-(\sigma_-) \ . \tag{71}$$

However, as for the bosonic string, the equations of motions still have to be supplemented by constraints, this time arising from the equations of motion of γ_{ab} and $\chi_{a\alpha}$. We will come back to these constraints in a moment.

Let us first discuss the issue of boundary conditions (focusing only on the closed string). Variation of the action leads to

$$(\psi_-^\mu \delta\psi_{\mu-} - \psi_+^\mu \delta\psi_{\mu+})\big|_{\sigma=0} = (\psi_-^\mu \delta\psi_{\mu-} - \psi_+^\mu \delta\psi_{\mu+})\big|_{\sigma=2\pi} \ . \tag{72}$$

In contrast to the bosonic coordinates X^μ, the fermions do not need to be periodic, they could also be antiperiodic, i.e.

$$\psi_+^\mu(\tau, \sigma = 2\pi) = \pm\psi_+^\mu(\tau, \sigma = 0) \quad , \quad \psi_-^\mu(\tau, \sigma = 2\pi) = \pm\psi_-^\mu(\tau, \sigma = 0) \ . \tag{73}$$

This leaves four different combinations of periodicity for left- and right moving fermions, each of which would solve the boundary conditions (72). Periodic boundary conditions are also called *Ramond* or just *R* boundary conditions, whereas antiperiodic boundary conditions are also called *Neveu-Schwarz* or just *NS* boundary conditions. Thus, the RNS string has four different sectors

R-R, NS-NS and R-NS, NS-R,

distinguished by the boundary conditions for the left - and right-moving fermions. Without going into any details, we just mention that the full string theory contains states from all of these sectors. This is required by consistency of the theory at the loop level. One can show now that the states in the R-R and NS-NS sectors correspond to space-time bosons, whereas those in the R-NS and NS-R sectors behave like space-time fermions. This is good news, as the absence of fermions in the bosonic string was one motivation for introducing the superstring. It turns out, however, that the spectrum still contains a tachyon and the theory is still not space-time supersymmetric, despite the presence of the fermions. Both of these problems are actually remedied by demanding consistency of the full interacting string theory. This requires a projection of the spectrum, the so called *GSO-projection* (named after Gliozzi, Scherk and Olive), for which there are two different versions, both of which remove the tachyon and render the spectrum supersymmetric.

Let us now come back to the constraints arising from the equations of motion of γ_{ab} and $\chi_{a\alpha}$. In light cone coordinates they are given by

$$0 \equiv T_{\pm\pm} = \frac{1}{2}(\partial_\pm X \cdot \partial_\pm X + i\psi_\pm \cdot \partial_\pm \psi_\pm) \quad , \quad 0 \equiv T_{F\pm} = \frac{1}{2}\psi_\pm \cdot \partial_\pm X \ . \tag{74}$$

As for the bosonic string, these constraints can either be imposed before or after quantization. Ultimately, the presence of these constraints is due to the fact that going to the superconformal gauge does not fully fix the gauge freedom. One can solve the constraints by making use of the remaining gauge symmetries, which are the conformal transformations discussed before in the context of the bosonic string, and the following local supersymmetry transformations:

$$\delta_\epsilon X^\mu = i(\epsilon^+\psi_+^\mu + \epsilon^-\psi_-^\mu) \quad , \quad \delta_\epsilon \begin{pmatrix} \psi_+^\mu \\ \psi_-^\mu \end{pmatrix} = -\begin{pmatrix} \epsilon^+\partial_+ X^\mu \\ \epsilon^-\partial_- X^\mu \end{pmatrix} \ , \tag{75}$$

with $\epsilon^\pm = \epsilon^\pm(\sigma_\pm)$.

The remaining local supersymmetry transformations allow one to choose the analogue of light cone gauge for the superstring, by setting $\psi_\pm^+ = 0$. Plugging this into

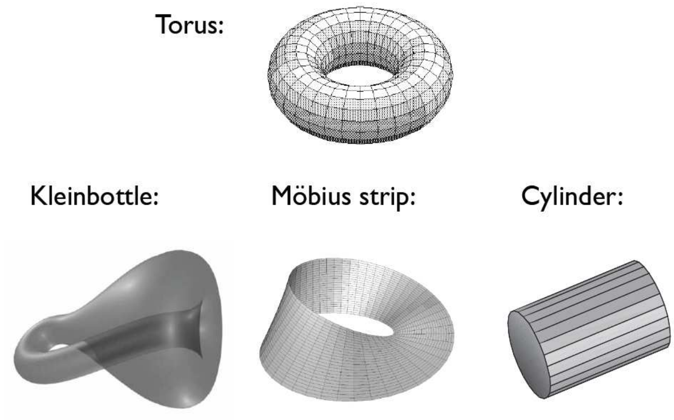

Figure 7: 1-loop diagrams in type I string theory. When calculating scattering amplitudes, one still has to attach the external strings.

the second constraint in (74), one sees that one can solve $T_{F\pm} = 0$ by expressing ψ^- in terms of the other ψ^I. This would be the starting point for the light cone quantization of the superstring.

As already mentioned, there are two types of GSO projections and, consequently, two different consistent superstring theories with only closed strings, the *type IIA* and *type IIB* theory. Among others, they also differ in their massless spectra. To illustrate this, we just mention the differences in the massless bosonic fields. They are given by

$$\begin{aligned} &\mathrm{NSNS}: g_{\mu\nu}, B_{\mu\nu}, \phi \quad , \qquad \mathrm{RR}: C_\mu, C_{\mu\nu\rho} \qquad (\mathrm{IIA})\ , \\ &\mathrm{NSNS}: g_{\mu\nu}, B_{\mu\nu}, \phi \quad , \qquad \mathrm{RR}: C, C_{\mu\nu}, C_{\mu\nu\rho\sigma} \qquad (\mathrm{IIB})\ . \end{aligned} \tag{76}$$

Obviously, the fields from the NSNS-sector coincide but the two theories differ in the RR-sector. The type IIA theory has a 1-form and a 3-form potential, whereas the type IIB theory has a scalar, a second 2-form and a 4-form potential, whose 5-form field strength is self-dual.

Apart from the type IIA and type IIB theories which only contain closed strings, there is another supersymmetric string theory which contains also open strings. This is called *type I*, as it has half as much supersymmetry. Apart from containing open strings, it is distinguished by the fact that its strings are non-oriented. Thus, apart from the torus (which already appeared in the oriented bosonic string, cf. fig.4) , there are three more 1-loop world-sheets, the Kleinbottle, the Möbius strip and the cylinder, cf. figure 7.

Apart from that, there are two supersymmetric string theories, which combine the left moving bosonic string with the right moving superstring. This leads to the so called *heterotic strings*, which come in two versions, differing for instance in the gauge group. A list of all the different consistent supersymmetric string theories can be found in table 1. All of them live in ten dimensions and the massless modes are, at low energies, described by supergravity theories.

Table 1: Different 10-dimensional superstring theories.

	open/closed	oriented	SUSY	Gauge group
IIA	closed	yes	$\mathcal{N}=2$	$U(1)$
IIB	closed	yes	$\mathcal{N}=2$	-
Heterotic	closed	yes	$\mathcal{N}=1$	$E_8 \times E_8$, $SO(32)$
I	both	no	$\mathcal{N}=1$	$SO(32)$

3. 1. FLUX COMPACTIFICATION

In this final section we would like to describe in a bit more detail one of the main topics of recent research – the topic of *flux compactification*.

In order to understand the motivation for flux compactification, let us revisit the example of the circle comactification that we already discussed above. To keep the discussion simple, we will start with a 5-dimensional action instead of a 26- or 10-dimensional one. Furthermore, we consider a simple 5-dimensional theory, which only consists of an Einstein-Hilbert term, i.e.

$$S_5 = \frac{1}{2\kappa_5^2}\int d^5x\sqrt{-g}R = \frac{1}{2\kappa_5^2}\int_{0\leq\theta\leq 2\pi r} d^4x\sqrt{-g}R \ . \tag{77}$$

Here we assumed a space-time of the form $M_5 = M_4 \times S^1$ in the second step, with a circle of radius r.

The 5-dimensional metric splits into three different fields, one which transforms as a symmetric tensor from the 4-dimensional point of view, one as a vector and one as a scalar. Schematically, one has

$$g_{MN} = \left(\begin{array}{cc} g_{\mu\nu} & A_\mu \\ A_\nu & \rho \end{array} \right) \ . \tag{78}$$

All three of these fields can be expanded in a Fourier series along the circle, i.e.

$$\mathcal{F}(x,\theta) = \sum_{n=-\infty}^{\infty} \mathcal{F}^{(n)} e^{in\theta/r} \quad , \quad \mathcal{F} \in \{g_{\mu\nu}, A_\mu, \rho\} \ . \tag{79}$$

Plugging this expansion, together with (78), into (77), one obtains an effective 4-dimensional action with an infinite number of fields. Interestingly, the dynamics of the zero modes in the Fourier expansion (79), i.e. the terms with $n=0$, is described by a 4-dimensional Einstein-Hilbert term coupled to a Maxwell field and a massless scalar, i.e.

$$\begin{aligned} S_4 \quad = \quad & \frac{1}{2\kappa_4^2}\int d^4x\sqrt{-g^{(0)}}\Big[\Big(R(g^{(0)}) - \frac{\rho^{(0)}}{4}F^{(0)}_{\mu\nu}F^{(0)\mu\nu} - \frac{1}{(\rho^{(0)})^2}\partial_\mu\rho^{(0)}\partial^\mu\rho^{(0)}\Big) \\ & + \Big(\nabla_\mu\rho^{(1)}\nabla^\mu\rho^{(1)} + \frac{1}{r^2}(\rho^{(1)})^2 + \ldots\Big)\Big] \ . \end{aligned} \tag{80}$$

The possibility to obtain 4-dimensional gravity coupled to electromagnetism in this way, i.e. by starting from a purely gravitational theory in 5 dimensions and expanding it around a product space-time with a circle direction, was the original motivation of *Kaluza* and *Klein*, who suggested this mechanism long before the advent of string theory.

However, there are two obvious drawbacks of this approach, challenging its phenomenological viability. First, there are infinitely many additional fields, the Fourier coefficients with $n \neq 0$. For illustration we kept in (80) the terms describing the dynamics of the scalar with $n = 1$. It is charged under the gauge field $A_\mu^{(0)}$ (hence the covariant derivatives) and has a mass term with $m^2 \sim r^{-2}$. This is a general feature and one can show that $g_{\mu\nu}^{(n)}, A_\mu^{(n)}$ and $\rho^{(n)}$ have masses $\sim \frac{|n|}{r}$. This is also consistent with the fact that we found $\rho^{(0)}$ to be a massless scalar. The mass of the higher Fourier modes arises from the momentum in the internal S^1-direction and it is familiar from our discussion of strings moving on a circle, cf. the first term in the second line of (63).

The tower of massive Kaluza-Klein modes could still be dealt with by assuming a very small radius of the internal circle. All the masses of the higher modes would be very large and they might have escaped observation in this way. However, the presence of the massless scalar (whose vacuum expectation value determines the radius of the circle), poses a serious problem for phenomenology. There are simply no massless scalars observed in nature. They would mediate an additional long-range force and they would also pose a problem for early universe physics, as their oscillations could easily overclose the universe. To make things worse, it is obvious from (80) that the expectation value of $\rho^{(0)}$ determines the gauge coupling of the Maxwell field and, thus, its variation in time and space is highly constraint. This requires to fix this value dynamically via a potential which is, however, not present in (80).

It turns out that the presence of massless scalars in the effective theory arising via compactification from higher dimensions is a general feature which is also omnipresent in string theory. One way to reconcile the fact that superstrings live in 10-dimensions and the observation of only 4 (large) dimensions, is to assume a space-time of the product structure $M_{10} = M_4 \times K_6$, with the 6-dimensional compact space K_6 playing the role of the circle in our example above. In the context of string theory, a particular class of compact spaces is distinguished, as it leaves some supersymmetry intact in the effective 4-dimensional theory. These K_6 are called *Calabi-Yau spaces*.

While the geometry of the S^1 is determined by a single parameter, its radius, the geometry of K_6 is much more complicated and, in general, described by a large set of parameters, each of which would give rise to a massless field in the effective theory. This is similar to the radius, which is given as the vacuum expectation value of the massless field $\rho^{(0)}$. In contrast to S^1, the higher dimensional spaces K_6 have two types of parameters, describing either their shape or their size (or sizes of their subspaces). The easiest example to exemplify these two types of parameters is the 2-dimensional torus, cf. fig. 8.

The massless fields arising from the parameters of the internal compactification space are called *moduli* and their presence in the effective 4-dimensional theory is known as the *moduli problem* of string theory. Flux compactification is a recent attempt to cope with this problem. It makes use of the fact that the spectrum of string theory also contains p-form fields with $(p+1)$-form field strengths, cf. (76). In 10 dimensions they have a kinetic term similar to a Maxwell-term, i.e. (for the case

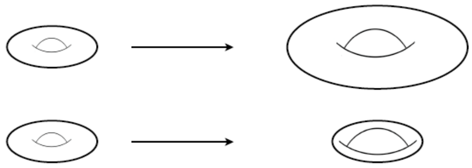

Figure 8: The moduli of a torus.

of $p = 2$)

$$\int d^{10}x\sqrt{-g}F_{IJK}F_{LMN}g^{IL}g^{JM}g^{KN} \ . \tag{81}$$

Now, as the internal components of the metric, g_{ij}, determine the moduli fields (similar to the internal component $g_{55} = \rho$ in the S^1-example), the kinetic term (81) would induce a potential in the effective 4-dimensional action, if the internal components of the 3-form field strength had a non-vanishing expectation value (called a flux), i.e. $\langle F_{ijk}\rangle \neq 0$. This is because the energy stored in the field F_{ijk} is minimized for a certain geometry of the internal space.

We do not want to go into further details of flux compactifications in string theory, but we would like to end this review by illustrating this effect within the 5-dimensional toy example with which we started this section. The simplest analog of the kinetic term (81) would be a kinetic term for a scalar in 5 dimensions, i.e. for a 0-form,

$$S_5 = \frac{1}{2\kappa_5^2}\int d^4x d\theta\sqrt{-g}(R - \partial_M\phi\partial^M\phi) \ . \tag{82}$$

Similar to (79), also ϕ has to be expanded in a Fourier series and gives rise to another tower of Kaluza-Klein fields. So far we did not gain anything yet. Focusing on the zero mode, $\phi = \phi^{(0)}(x) + \ldots$, one would just end up with another massless scalar in 4 dimensions and the 4-dimensional action (80) would get the additional term

$$\delta S_4 \sim -\frac{1}{2\kappa_4^2}\int d^4x\sqrt{-g^{(0)}}\partial_\mu\phi^{(0)}\partial^\mu\phi^{(0)} + \ldots \ . \tag{83}$$

However, in order to employ the strategy outlined above, we would now like to consider a non-vanishing flux for the internal components of the field strength of the scalar, i.e. for $\partial_\theta\phi$. This can be done by modifying the above expansion ansatz according to

$$\phi = \phi^{(0)}(x) + m\theta + \ldots \ . \tag{84}$$

Plugging this ansatz into (82), one ends up with

$$\delta S_4 \sim -\frac{1}{2\kappa_4^2}\int d^4x\sqrt{-g^{(0)}}\Big((\rho^{(0)})^{-1}m^2 + (\partial_\mu\phi^{(0)} - mA_\mu^{(0)})(\partial^\mu\phi^{(0)} - mA^{(0)\mu})\Big) \ . \tag{85}$$

This result is interesting for several reasons. First, the flux induced a potential for the scalar $\rho^{(0)}$. Even though the potential is not very useful in the case at hand, as it shows a runaway behavior (it is minimized at $\rho^{(0)} \to \infty$), it illustrates the general principle mentioned above. In the context of string theory, more complicated potentials arise that can lead to stable minima for the moduli fields at finite vacuum expectation values. Also the problem with the second massless scalar $\phi^{(0)}$ is solved in a way that occurs often in flux compactifications. Its coupling to the gauge field represents a gauge-invariant mass term for $A_\mu^{(0)}$ (as $\phi^{(0)}$ could be gauged away). Thus, it becomes the longitudinal component of a massive vector field and does not appear as a massless scalar anymore. This mass term of the vector field reflects the fact that the shift symmetry along the circle direction is broken via the ansatz (84).

This brings our short introduction to an end and for further studies we refer to the following short guide to the literature.

4. GUIDE TO THE LITERATURE

Below, there is an incomplete list of references about string theory and related matters. The first seven are available on the net, whereas the others are text books. More advanced topics like D-branes or the AdS/CFT correspondence are covered in Polchinski's lecture notes and the review by D'Hoker and Freedman, respectively. The text books comprise different levels of difficulty. The most easily accessible ones are the ones by Mahon and Zwiebach. The latter one almost exclusively concentrates on the bosonic string, whereas the former one gives a broad introduction. It leaves out many details which can be found in the other text books, but it gives a fast overview over the field. Green, Schwarz and Witten is still the classic introduction into more mathematical aspects of string compactifications, Lüst and Theisen nicely covers the CFT aspects of string theory, Johnson contains many details on D-branes not discussed in the other text books and the books by Becker, Becker and Schwarz and by Kiritsis can be understood as updates on the classic book by Polchinski, covering also more recent developments like the AdS/CFT correspondence and flux compactifications.

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