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FORMATION OF PLANETS IN GASEOUS VORTICES AND THE LAW OF PLANETARY DISTANCES

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Abstract. The protoplanetary nebula probably passed through the stage of an accretion disk, which contained the gaseous and dust components. Its evolution led to the formation of planetesimals, which eventually accumulated into the planets. Due to the difficulties of the 'standard' model of the Solar System formation, there has been a revival of the theories of the 'persistent gaseous vortices' recently. The assumption that planets formed near the centers of the vortices in one of the theories (Barge and Sommeria 1995) yields an implicit law of planetary distances. Within the accepted range $1/2 \leq q \leq 3/4$ of the main parameter of the nebula (the inverse power of the radial temperature dependence), the law predicts an increase of the planetary distances that is considerably stronger than linear on the *logarithmic* scale. An acceptable fit to the observed distances can be obtained only for $q = 3/4$ and if Pluto is excluded. An explicit but approximate planetary distance law shows a misleadingly good fit.

1. INTRODUCTION

There is a great variety of theories of the Solar System formation. However, there is a broad agreement that the planets have formed from dust grains embedded into a gaseous disk – the Protoplanetary Nebula. The Nebula probably passed through a stage of a turbulent accretion disk. The dominant mechanisms of accretion are the turbulent diffusion for small particles and the gravitational binding for large particles. For intermediate size particles – some 3-5 orders of magnitude by diameter – neither mechanism is efficient. Hence at least two distinct stages of the solid-gas phase decoupling occurred: (i) settling of the dust grains towards the mid-plane of the gaseous disk, (ii) gravitational collapse of the dust sub-disk into kilometer-sized planetesimals.

The subsequent evolution of the Nebula is relatively well-understood within the so-called Standard Model (Safronov 1969, Wetherill 1980). However, there are two major difficulties faced by this scenario (Barge and Sommeria 1995): (1) the time scale for the formation of the embrios of the giant planets may be too long to allow the capture of the sufficient amount of the remaining nebular gases; (2) the dust sub-disk may not have reached the density required for the gravitational instability.

Motivated by the difficulties of the Standard Model and by an improved theoretical understanding of turbulence in astrophysical disks, the theories of 'persistent gaseous vortices' have been revived recently (Barge and Sommeria 1995, Chavanis 2000).

Historically, the persistent gaseous vortices are connected with (neo-)Cartesian theories of the origin of the Solar System. The first theory of persistent vortices that included gravity was proposed by Weizsäcker (1944). Initially, the Weizsäcker's theory was received enthusiastically (Chandrasekhar 1946), but was soon abandoned because the vortices would dissipate before any condensation could take place (Williams and Cremin 1968). Perhaps the most remarkable feature of the Weizsäcker's theory was an explanation of the geometric progression of the planetary distances:

$$r_n = r_0 \, b^n, \tag{1}$$

where r_0 and b are constants. The exponential form of the Titius-Bode law had attracted considerable attention before (Nieto 1972, Ignjatović 2003), but Weizsäcker's explanation was the first one to gain a broader acceptance. Graner and Dubrulle (1994) showed that the formula (1) is easily derivable in any scale-invariant and axially symmetric model regardless of its physical content.

According to many authors, a successful theory of the origin of the Solar System should account for the non-random distribution of planetary orbits. Most astronomers and astrophysicists dispute the significance of planetary distance laws like (1) (Hayes and Tremaine 1998), but this is still a matter of debate (Lynch 2003). Today, the Titius-Bode-like laws are disfavored largely because the planetary distances (may) have changed after the early stages of the Solar System formation due to 1) the instability of orbits in the inner Solar System over the age of the System, and/or 2) giant planet migration during the later stages of their formation.

Barge and Sommeria (1995) claim that their theory of Solar System formation within persistent gaseous vortices (BS model) is consistent with an approximately geometric spacing of the planetary orbits. Bearing in mind the possible variability of the planetary distances, it is still interesting to investigate if the BS model is consistent with the law of planetary distances (1) and with the observed planetary distances.

In this paper we adopt the basic assumptions of the BS model. We derive a recursive relation for planetary distances that follows from BS theory and compare the predicted distances to the law (1) and to the observed planetary distances.

2. THE BARGE-SOMMERIA MODEL

The BS model aims to overcome the major difficulties of the second stage in the development of the solid material in the Protoplanetary Nebula. According to the model, the solid particles, once settled in the Nebula mid-plane, are captured and concentrated into long-lived vortices.

Unlike three-dimensional eddies, which are quickly damped, two-dimensional turbulence can persist without energy dissipation. Under the standard assumption of the hydrostatic equilibrium, the dynamical flow is two- dimensional. Small vortices arising in the flow have a subsonic speed. The vortices grow by merging until they reach the size of the order of the local thickness H of the Nebula, $R \sim H$, when they reach the local speed of sound and thus dissipation due to sound waves sets in. Barge and Sommeria argued that such vortices could persist for some 500 rotation periods, but for smaller vortices the typical time scale is 50 rotation periods (Godon and Livio 2000). The solid particles are captured by the vortices, the capture mechanism being strongly dependent on the size of the particles. The capture rate reaches a maximum around 7.5 AU (which may be an indication of a later-stage migration of Jupiter).

The supply of the solid particles near a vortex orbit is maintained by the velocity difference between the gas and particles since the velocity of the gas is slightly smaller than Keplerian velocity due to the radial pressure gradient. Inside the vortices, the BS model predicts that the surface density increases by several orders of magnitude triggering the gravitational instabilities and leading to the formation of planetesimals. The details of the mechanism responsible for the formation and maintenance of the vortices are beyond the scope of this paper.

At a given distance from the Sun, only very few vortices are expected to remain. However, vortices with sufficient radial separation – a few times their size – can no longer merge. Thus we can expect "a set of independent isolated vortices with radial separation scaling like the nebula thickness" (Barge and Sommeria 1995). Therefore, $r_{n+1} - r_n \propto H_n$, where H_n is the nebula thickness at r_n. We will refine this relationship and take it as one of our assumptions. Beforehand, we present a standard and elementary derivation of the scaling law of $H(r)$ as this law is crucial for our analysis.

The disk is assumed to be nearly Keplerian (the rotation velocity of the gaseous component being somewhat smaller than the Keplerian velocity); the self-gravity of the nebula is neglected. The proto-Sun is assumed to have almost acquired the mass $M_\odot$. The condition of the hydrostatic equilibrium in the axial direction is given by:

$$\frac{\partial p}{\partial z} = -\frac{GM_\odot\,\rho(r,z)}{(r^2+z^2)^{3/2}}\,z, \tag{2}$$

where G is the gravitational constant and $\rho(r,z)$ density of the gas. We assume that the nebular gas is ideal so that the equation of state is

$$p = \frac{R}{\mu}\,\rho T, \tag{3}$$

where μ is the molecular mass of the gas, assumed to be constant.

A further assumption is that the disk is thin, $z \ll r$. This approximation will be justified later. Under these simplifications the Eqs. (2) and (3) yield

$$\frac{1}{p}\,\frac{\partial p}{\partial z} = -\,GM_\odot\,\frac{\mu}{R}\,\frac{1}{T(r,z)}\,\frac{z}{r^3}. \tag{4}$$

Another standard assumption is that the z-component of the temperature gradient can be neglected: $T(r,z) = T(r,0)$ so that from (4) we obtain

$$p(r,z) = p(r,0)e^{-z^2/d^2}, \tag{5}$$

where

$$d = \sqrt{\frac{2}{GM_\odot}\,\frac{R}{\mu}\,T}\,r^{3/2} \tag{6}$$

is the local 'half-thickness' of the disk.

The radial temperature dependence, or 'temperature law', is usually assumed to have the form of a power law (Cuzzi et al. 1993):

$$T(r,0) = T_0\Big(\frac{r_0}{r}\Big)^q, \tag{7}$$

where $T_0 = T(r_0, 0)$, r_0 being a reference radius, *e. g.* 1 AU. Certain observational and theoretical considerations appear to constrain the exponent to the interval $1/2 \leq q \leq 3/4$ (Cuzzi et al. 1993). We have accepted the power law (7) here, but its applicability to the problem of the planetary distances needs to be investigated.

From (6) and (7) we find the scaling law for the thickness of the disk:

$$d = d_0\Big(\frac{r}{r_0}\Big)^{\frac{3}{2}-\frac{q}{2}} = d_0\Big(\frac{r}{r_0}\Big)^k, \tag{8}$$

where

$$d_0 = \sqrt{\frac{2}{GM_\odot}\frac{R}{\mu}T_0}\, r_0^{3/2}. \tag{9}$$

The realistic choice is $\mu \approx 2$ (mainly molecular hydrogen) and $T_0 \approx 300K$ at $r_0 = 1$ AU, which gives $d_0 \approx 0.04$ AU.

In BS Model the choice of the temperature law exponent is $q = 1/2$. We will consider two values, namely 1/2 and 3/4, without pretending to assign a particular meaning to these values in terms of the physical parameters of the model. Notice that according to (8), for any choice of the exponent q within the above interval, the scaling of the thickness of the disk with the radial coordinate is considerably stronger than linear.

Let us return to the justification of the usual thin-disk approximation. The correction to scaling law (8) due to finite thickness of the disk would go like $3d^2/r^2 \propto (d_0/r_0)^2$. Even for $q = 1/2$ at the edge of the disk ($r \sim 50$ AU) the correction amounts to less than 1%. The other corrections for known effects, such as those due to real gas equation of state, are not expected to exceed 1% either. However, this does not include the uncertainties connected with the power law (7).

3. THE PLANETARY DISTANCES

The size of the vortices scales like the thickness of the nebula:

$$R \sim H = 2d \propto \Big(\frac{r}{r_0}\Big)^k. \tag{10}$$

The result (10) *per se* does not lead to a law of planetary distances since, in principle, any distribution of the vortices can be consistent with (10) provided the vortices are not too close to each other. However, in BS Model the radial interval between the vortices scales like the thickness of the nebula, $r_{n+1} - r_n \propto H_n$. From this and the scaling law (10) Barge and Sommeria conclude: "this is consistent with an approximate geometric progression of the planetary positions", implying a recursive relation like

$$r_{n+1} - r_n = c\, r_n^k, \tag{11}$$

where c is constant and k is close to 1.

Chavanis (2000) agrees with the conclusion about the approximate geometric progression. What they have apparently missed is that the power law (10) for $k \geq 9/8$ will deviate rather appreciably from the linear case (the one leading to the geometric progression) if applied to the distances stretching over two orders of magnitude.

Table 1: Theoretical distances r_n compared to the observed distances $\bar{r}_n$. RMSR for relative residuals is shown. The dwarf planet distances (in the parentheses) were not used in the fits.

n	$\bar{r}_n$ (AU)	r_n from (1)	r_n from (13)		r_n from (18)	
		$k=0$	$k=5/4$	$k=9/8$	$k=5/4$	$k=9/8$
1	0.387	0.349	0.497	0.422	0.413	0.392
2	0.723	0.607	0.692	0.647	0.635	0.626
3	1.000	1.054	0.996	1.018	1.010	1.025
4	1.524	1.831	1.488	1.649	1.662	1.717
5	(2.77)	3.181	2.334	2.758	2.828	2.943
6	5.203	5.526	3.893	4.794	4.977	5.163
7	9.540	9.599	7.062	8.711	9.061	9.269
8	19.19	16.68	14.49	16.70	17.06	17.03
9	30.07	28.97	36.52	34.14	33.22	32.01
10	(39.46)	50.32	157.1	75.75	66.91	61.59
RMSR		0.113	0.200	0.097	0.083	0.081

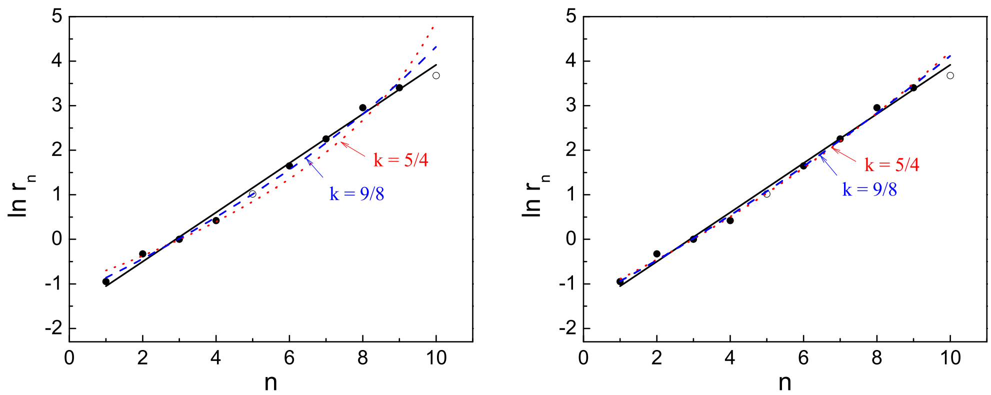

Figure 1: The fits of (1) & (13) (left) and (1) & (18) (right) to the observed distances.

Instead of the simplified relation (11) we will derive a relation that, in our view, better describes the geometry of the Nebula, but our qualitative conclusion will remain valid. Suppose the size of the 'cleared' zone around each vortex is proportional to the size of the vortex: $l_n = C\,R_n$. The interval between the centers of two neighboring vortices is divided by the two corresponding zones:

$$r_{n+1} - r_n = l_{n+1} + l_n = C(R_{n+1} + R_n). \tag{12}$$

On the other hand, (10) implies $R_n \propto r_n^k$ so that (12) gives

$$r_{n+1} - r_n = \bar{c}\,(r_{n+1}^k + r_n^k), \tag{13}$$

where $\bar{c}$ is a constant. While (13) is not a necessary consequence of the BS Model it is probably the most natural choice. The nonlinear recursive relation (13) can be fitted to the observed planetary distances either numerically or by expanding in k around 1.

1) The results of numerical fits to the relation (13) are shown in Table 1 and in Fig. 1. The fit for $k = 5/4$ is obviously poor as seen from the large root mean square residual (RMSR) for relative residuals, but the fit for $k = 9/8$ is better than the exponential law (1). The fit parameters for $k = 5/4$ $(9/8)$ are $\bar{c} = 0.187$ (0.227) and $r_0 = 0.366$ (0.281). The fits that include Pluto are much worse and are not shown.

2) In order to gain some insight it may be useful to derive an approximate planetary distance law. First we divide relation (13) by r_n and then divide the resulting equation with the same relation with n replaced by $n-1$ to obtain

$$\frac{\alpha_{n+1}^k + 1}{\alpha_{n+1} - 1} = \alpha_n^{1-k} \frac{\alpha_n^k + 1}{\alpha_n - 1}, \tag{14}$$

where $\alpha_n \equiv r_n/r_{n-1}$. Let us expand (14) by taking $\varepsilon = k - 1$ to be a parameter: $\alpha_n = b + f_n \varepsilon$, where b is the constant ratio that corresponds to $\varepsilon = 0$. To the first order in ε equation (14) yields an arithmetic progression:

$$f_{n+1} - f_n \approx \frac{1}{2}(b^2 - 1)\ln b, \tag{15}$$

where $f_0 = 0$ by assumption so that

$$\alpha_n \approx b + n\,\varepsilon \frac{b^2-1}{2} \ln b = b\left(1 + n\,\varepsilon \frac{b^2-1}{2b} \ln b\right). \tag{16}$$

Taking ln of the identity $r_n = r_0\,\alpha_1 \cdot \ldots \cdot \alpha_n$ and using $\ln(1+x) \approx x$ we get from (16)

$$\ln r_n \approx \ln r_0 + n \ln b + \varepsilon \frac{b^2-1}{2b} \ln b \sum_{i=1}^{n} i = \ln r_0 + n \ln b + n(n+1)\,\varepsilon \frac{b^2-1}{4b} \ln b, \tag{17}$$

i. e.

$$r_n \approx r_0\, b^{n + n(n+1)\,\varepsilon \frac{b^2-1}{4b}}. \tag{18}$$

The fit parameters for (1) are $r_0 = 0.201$, $b = 1.737$ while for (18) with $k = 5/4$ $(9/8)$ are $r_0 = 0.277$ (0.251), $b = 1.438$ (1.527). Comparing the fits of (18) to the fits of (13), shown in Table 1 and in Fig. 1, we can see that (18) reproduces the rapid increase of the distances with n. However, the good fit is misleading, especially for $k = 5/4$, as the errors ($\sim i^2\varepsilon^2$) in (17) sum up and rapidly become large beyond certain n.

4. CONCLUSIONS

We have investigated if the planetary distances predicted by the BS Model are consistent with the observed distances. The correlation is reasonably good only if the temperature law of the Protoplanetary Nebula goes like $r^{-3/4}$. The fits that include Pluto are very poorly correlated with the observed distances. An approximate planetary distance law obtained from the same model shows an empirically good fit, but this is misleading as the approximation is not valid for the outer Solar System. Our analysis suggests that fitting the observed planetary distances to a *physical model* of the Solar System formation may lead to certain constraints on the model, but one should not be misled by the usually simple, empirically successful planetary distance 'laws'.

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