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STATIONARY VORTICES IN GRAVITATING FLUID

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Abstract. *Nonlinear self-organization of perturbations in differentially rotating, nonuniform, gravitating systems is studied, and two types of stationary nonlinear solutions, in the form of tripolar vortices and vortex chains of gravitational potential and density, dependent on the spatial profiles of the basic state quantities, are found.*

We study perturbations propagating in differentially rotating, spatially nonuniform, self-gravitating fluid systems in the stage of highly developed nonlinearities, that can be described by the following simple set of equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0, \quad \nabla^2 \varphi = 4\pi G \rho, \quad \left(\frac{\partial}{\partial t} + \vec{v}\nabla\right)\vec{v} = 2\vec{v} \times \vec{\Omega}_0 - \nabla\varphi - \frac{\nabla p}{\rho}. \qquad (1)$$

Here $\Omega_0(x)\vec{e}_z$ is the vector of angular velocity, φ is the gravity potential, $p = c_s^2 \rho$, c_s is the isothermal sound speed, and other notation is standard. All basic state quantities depend on the x-coordinate only, and we introduce the fluid velocity describing differential rotation in the basic state $v_0(x)\vec{e}_y$. The subscript 0 is used to denote the basic state quantities. We shall study low frequency, $\partial/\partial t \ll \Omega_0$, two dimensional perturbations propagating perpendicularly to $\vec{\Omega}_0$. This approach is well satisfied for certain interstelar objects, for example molecular clouds with the period of rotation of approximately $7 \cdot 10^6$ years, which is only about 1/50 part of their average time of existence. In such circumstances slow modes investigated here have enough time to develop. In order to demonstrate the instability in Jeans' sence, it turns out that third order small terms $(\partial/\partial t)^3/\Omega_0^3$ in Eq. (1) are to be kept.

For the purpose of finding nonlinear stationary vortex solutions of the basic Eq. (1) it is enough to keep the second order linear and nonlinear small terms only. On that time scale the effects of eventual gravitational contraction will be assumed as small ones and will not be taken into account. In this sence also the pressure terms will be neglected in the future derivations, and we shall concentrate on finding coherent vortex structures. In the limit when the basic-state and perturbed velocities, $\vec{v}_0(x)$, $\delta\vec{v}$, are of the same order, from Eq. (1) we obtain the nonlinear equation that will be the basic one for the further study:

$$\left[\frac{\partial}{\partial t} + \vec{e}_z \times \nabla_\perp(\varphi + \psi) \cdot \nabla_\perp\right] \left(\log\rho_0 - \log\Omega_0 + \alpha\nabla_\perp^2\varphi - \alpha_0\nabla_\perp^2\psi\right) = 0. \qquad (2)$$

Here $\vec{v}_0(x) = (\vec{e}_z \times \nabla_\perp\psi)/(2\Omega_0)$, $\nabla_\perp = \vec{e}_x(\partial/\partial x) + \vec{e}_y(\partial/\partial y)$, $\alpha_0 = v_{00}/2l_0\Omega_0$, $\alpha = \alpha_0(4\Omega_0^2/\omega_0^2 - 1)$; l_0, v_{00} are some characteristic dimension and velocity of the system that is investigated, and the following normalization is introduced:

$$\frac{\partial}{\partial t} \to \frac{l_0}{v_{00}}\frac{\partial}{\partial t}, \quad \nabla_\perp \to l_0\nabla_\perp, \quad (\varphi, \psi) \to \frac{1}{2\Omega_0 l_0 v_{00}}(\varphi, \psi).$$

Eq. (2) is generic in problems dealing with nonlinear vortex structures driven by nonlinearities of the vector-product type in spatially nonuniform systems, and it can be solved using a procedure developed in our previously published papers dealing with quite different problems in plasma physics theory (Vranješ et al. 1998a, 1999b, Vranješ 1999a). We look for traveling solutions that are stationary in a reference frame moving with a velocity u in the direction of y-axis. In this case Eq. (2) can be integrated yielding:

$$\log\rho_0(x) - \log\Omega_0(x) + \alpha\nabla_\perp^2\varphi - \alpha_0\nabla_\perp^2\psi = F(\varphi + \psi - ux), \qquad (3)$$

where $F(\xi)$ is an arbitrary function of the given argument.

Tripolar vortex solutions: We proceed by choosing the functional $F(\xi)$ as linear one, i.e., $F(\xi) = F_0 + F_1 \cdot \xi$, and allowing for different values of the given constants $F_{0,1}$ inside and outside of an arbitrary circle of radius r_0. On condition when the basic state can be approximated by the following set of equations:

$$\psi(x) - ux = ax^2, \quad \frac{1}{\alpha}\log\rho_0 - \frac{1}{\alpha}\log\Omega_0 - \frac{\alpha_0}{\alpha}\nabla_\perp^2\psi = bx^2, \qquad (4)$$

Eq. (3) can be rewritten as:

$$(\nabla_\perp^2 - F_1)\left[\varphi - \frac{b - aF_1}{F_1}x^2 + \frac{F_0}{F_1} - \frac{2(b - aF_1)}{F_1^2}\right] = 0. \qquad (5)$$

According to the definition of $\psi(x)$, Eq. (4) yields a linear profile of the basic state shear flow. In polar coordinates (r, θ), Eq. (5) separates variables, and can be solved independently outside and inside of the circle. For vanishing perturbations at infinity we have $F_1^+ = b/a$ (here the superscript + denotes the outside values regarding the given circle), and the outside solution of the above equation can be written as:

$$\varphi^+(r, \theta) = b_0 K_0(\lambda_1 r) + b_2 K_2(\lambda_1 r)\cos 2\theta, \qquad (6)$$

where $F_0^+ = 0$, and we introduced $F_1^+ \equiv b/a = \lambda_1^2$, and $K_{0,2}(\lambda_1 r)$ are modified Bessel functions of the given order. Similarly, the inside solution is:

$$\varphi^-(r,\theta) = a_0 J_0(\lambda_2 r) - \frac{C_1 r^2}{2} - C_2 + \left[a_2 J_2(\lambda_2 r) - \frac{C_1 r^2}{2}\right]\cos 2\theta. \tag{7}$$

Here $J_{0,2}(\lambda_2 r)$ are Bessel functions, and

$$\lambda_2^2 = -F_1^-, \quad C_1 = \frac{b + a\lambda_2^2}{\lambda_2^2}, \quad C_2 = -\frac{2}{\lambda_2^2}\left(\frac{F_0}{2} + C_1\right).$$

The unknown constants in the solutions (6), (7) are to be found using the following physically justified continuity conditions at $r = r_0$: continuity of the potential φ and its derivative with respect to the coordinate r; the continuity of the function $F(\xi)$ to avoid singularities of the basic nonlinear equation at the circle, and we take the argument ξ as constant at $r = r_0$.

Now, from Eqs. (1), (6), (7), we may calculate the density ρ (in units of ρ_{00}):

$$\rho^+(r,\theta) = \frac{b}{a}\left[\beta_0 K_0(\lambda_1 r) + \beta_2 K_2(\lambda_1 r)\cos 2\theta\right], \quad r > r_0, \tag{8}$$

$$\rho^-(r,\theta) = -(a\lambda_2^2 + b)\frac{r^2}{2} - 2C_1 + \lambda_2^2\left[\frac{C_1 r^2}{2} - \alpha_0 J_0(\lambda_2 r)\right]$$
$$- \left[(a\lambda_2^2 + b)\frac{r^2}{2} + \lambda_2^2\left[\alpha_2 J_2(\lambda_2 r) - \frac{C_1 r^2}{2}\right]\right]\cos 2\theta, \quad r < r_0. \tag{9}$$

This is a tripole with a negative, rotating central part representing a rarefication of the background cloud density, and with two lateral, oppositely rotating positive vortices that can be subject to eventual further contraction in the Jeans' sence.

Vortex chain solutions: The function $F(\xi)$ in Eq. (3) can be chosen in other way, yielding another type of nonlinear stationary solutions. Let $F(\xi) = \xi + A\kappa^2 \exp(-2\xi/A)$, where A and κ are some constants, and assume a tanh profile of the fluid flow, i. e.,

$$\psi(x) = ux + A\log\cosh\kappa x, \quad \text{and} \quad \frac{\rho_0(x)}{\Omega_0(x)} = (\cosh\kappa x)^{A\alpha}.$$

After some algebra, Eq. (3) can be rewritten as:

$$(\nabla_\perp^2 - 1)\widehat{\varphi} - f(x)\left[\exp(-\widehat{\varphi}) + \frac{\alpha_0}{\alpha}\right] = 0, \quad \text{where} \quad f(x) = \frac{2\kappa^2}{\cosh^2\kappa x}, \quad \widehat{\varphi} = \frac{2\varphi}{A}. \tag{10}$$

Eq. (10) can be solved numerically in the following manner. We look for solutions consisting of a nonlinearly generated potential $\widehat{\varphi}_0(x)$ localized in the x-direction, and a wave-like perturbation periodic in the y-direction, i.e., $\widehat{\varphi}(x,y) = \widehat{\varphi}_0(x) + \widehat{\delta\varphi}(x)\cos ky$, where $|\widehat{\delta\varphi}(x)| \ll |\widehat{\varphi}_0(x)|$. Similar to Vranješ and Jovanović (1996), Vranješ (1998b, 1999c), we obtain the following set of equations for $\widehat{\varphi}_0(x)$, and $\widehat{\delta\varphi}(x)$:

$$\left(\frac{d^2}{dx^2} - 1\right)\widehat{\varphi}_0(x) - f(x)\left[\exp[-\widehat{\varphi}_0(x)] + \frac{\alpha_0}{\alpha}\right] = 0, \tag{11}$$

$$\left(\frac{d^2}{dx^2} - k^2 - 1\right)\widehat{\delta\varphi}(x) + f(x)\exp[-\widehat{\varphi}_0(x)]\,\widehat{\delta\varphi}(x) = 0. \tag{12}$$

By varying parameters κ, k, and looking for localized solutions in the x-direction, Eqs. (11), (12) are solved numerically. The contour plot of one particular solution for the gravity potential and for $\kappa \approx 0.4$, $k \approx 0.7$, reveals a single chain localized in the direction perpendicular to the basic state gradients and periodic along the flow. The corresponding expression for the perturbed density $\rho(x, y)$ can be readily written as:

$$\rho(x,y) = \widehat{\varphi}_0(x) + \left[\exp[-\widehat{\varphi}_0(x)] + \frac{\alpha_0}{\alpha}\right] f(x) + [1 - f(x)\exp[-\widehat{\varphi}_0(x)]]\,\widehat{\delta\varphi}_0(x)\cos ky.$$

It represents a triple chain of traveling vortices. The lateral chains have much less amplitude than the central hump, and are in fact rarefactions of the basic state density.

To conclude, starting from standard equations describing perturbations in a differentially rotating nonuniform gas cloud in a Cartesian geometry, we have derived a nonlinear equation comprising the vector-product type nonlinear term which is known to be responsible for creation of stationary vortex structures, both in plasmas and ordinary fluids. It is shown that, depending on the spatial profile of the fluid flow and on other variables describing the basic state, two qualitatively different types of solutions are possible. They are stationary, traveling in the direction of the flow and, on much longer time scales, can be subject to the gravitational collapse in Jeans' sence.

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