

THE EQUATION OF STATE OF A BOSE GAS: SOME ANALYTICAL RESULTS

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Abstract. This is a short review of the main results on the equation of state of a degenerate, non-relativistic Bose gas. Some known results are expressed in a new form, and their possible applications in astrophysics and solid state physics are pointed out.

1. INTRODUCTION

The aim of this coriting is to review briefly some results concerning the equation of state (EOS) of a non-relativistic, degenerate, Bose gas. The study to be reported is a logical continuation of previous work concerning the Fermi-Dirac integrals, and the EOS of a Fermi gas (Čelebonović, 1998, a,b). Apart from being useful pedagogically, this review contains new expressions of some existing results.

2. CALCULATIONS

It can be shown (for example Landau and Lifchitz, 1976) that the number density of a Bose gas is given by the following integral:

$$n = 3D \frac{N}{V} = 3D \frac{gm^{3/2}}{2^{1/2}\pi^2\hbar^3} \int_0^\infty \frac{\epsilon^{1/2}d\epsilon}{\exp[(\epsilon - \mu)/T] - 1} \quad (1)$$

All the symbols have their usual meanings, s is the particle spin, $g = 3D2s + 1$ and Boltzmann's constant has been set equal to 1.

Introducing the change of variables $\frac{\epsilon}{T} = 3Dz$, it follows from Eq. (1) that

$$n = 3D \frac{g(mT)^{3/2}}{2^{1/2}\pi^2\hbar^3} \int_0^\infty \frac{\sqrt{z}dz}{\exp[z - \frac{\mu}{T}] - 1} \quad (2)$$

The function under the integral can be transformed as follows

$$\frac{\sqrt{z}}{\exp[z - \frac{\mu}{T}] - 1} = 3D \frac{\sqrt{z}}{\exp[z - \frac{\mu}{T}] (1 - \exp[\frac{\mu}{T} - z])} \quad (3)$$

Developing Eq. (3) into series, one gets the expression for the integral in Eq. (2)

$$I_B = 3D \int_0^\infty \frac{\sqrt{z} dz}{\exp\left[z - \frac{\mu}{T}\right] - 1} = 3D \sum_{l=3D0}^\infty \int_0^\infty \sqrt{z} \exp\left[(l+1)\left(\frac{\mu}{T} - z\right)\right] dz \quad (4)$$

which after some algebra can be transformed into the following final form

$$\begin{aligned} I_B &= 3D \sum_{l=3D0}^\infty \exp\left[(l+1)\frac{\mu}{T}\right] \int_0^\infty \exp[-(l+1)z] dz = \\ &= 3D \sum_{l=3D0}^\infty \exp\left[(l+1)\frac{\mu}{T}\right] \frac{\sqrt{\pi}}{2} (l+1)^{-3/2} \end{aligned} \quad (5)$$

Generalizing the reasoning which led to Eq. (5), it can be shown that

$$I_n = 3D \int_0^\infty \frac{z^n dz}{\exp\left[z - \frac{\mu}{T}\right] - 1} = 3D \sum_{l=3D0}^\infty \exp\left[(l+1)\frac{\mu}{T}\right] \frac{\Gamma(n+1)}{(l+1)^{(n+1)}} \quad (6)$$

where Γ denotes the gamma function. Inserting Eq. (5) into Eq. (1), one gets the following form of the EOS of a Bose gas

$$n = 3D \frac{g(mT)^{3/2}}{2^{1/2} \pi^2 \hbar^3} \sum_{l=3D0}^\infty \frac{\sqrt{\pi}}{2} \exp\left[(l+1)\frac{\mu}{T}\right] (l+1)^{-3/2} \quad (7)$$

This EOS relates the chemical potential, temperature and number density of a Bose gas. Inserting $\mu = 3D0$ in Eq. (7), one can determine the temperature T_B of Bose condensation. It thus turns out that

$$T_B = 3D \frac{2\hbar^2}{\pi m} \left(\frac{n}{g\zeta(3/2)} \right)^{2/3} \quad (8)$$

where $\zeta(3/2)$ denotes Riemann's zeta function.

Usually, the EOS of a system is a relationship between its pressure, volume and energy. It is known from general statistical physics (such as Landau and Lifchitz, 1976) that for a Bose gas the EOS has the form

$$pV = 3D \frac{2}{3} E \quad (9)$$

where V is the volume and E the energy of the system. The expression for the energy of a Bose gas contains an integral of the form given by Eq. (6) with $n = 3D3/2$ and the same prefactor as the right side of Eq. (1). Using Eqs. (2), (6) and (9), the following final result is obtained for the EOS of a Bose gas in the region $T \succ T_B$.

$$p = 3D \frac{2^{1/2} g}{3\pi^2} \left(\frac{mT}{\hbar} \right)^{3/2} \sum_{l=3D0}^\infty \exp\left[(l+1)\frac{\mu}{T}\right] \frac{\Gamma(5/2)}{(l+1)^{5/2}} \quad (10)$$

In the domain $T \prec T_B$, for which $\mu = 3D0$ one gets that

$$p = 3D \frac{2^{1/2} g}{3\pi^2} \left(\frac{mT}{\hbar} \right)^{3/2} \Gamma(5/2) \zeta(5/2) \quad (11)$$

3. DISCUSSION

In the preceding section we have derived several analytical expressions for the EOS of a Bose gas, both in the region of the existence of the Bose condensation and outside it. The derivation of these expressions (and even more their extension to the $T = 3D0K$ case) was an interesting problem in itself. However, much more interesting are the possibilities for applications of the results obtained here in studies of various systems occurring in astrophysics and physics.

Bose gas occurs in astrophysics in interesting and varied situations, which range from the early universe to the interior of the giant planets. In cosmology, Bose gases occur during the reheating after inflation. In a recent paper (Khlebnikov and Tkachev, 1999) a nonequilibrium Bose gas with an attractive interaction between particles was studied. It was shown there that the system evolves into a state that contains drops of the Bose-Einstein condensate. Could it be the possible explanation of the formation of primordial clumps out of which later clusters of galaxies were formed? In planetary interiors, of course depending on the chemical composition, Bose gas can occur in some cases. Closely related is the problem of the insulator \rightarrow metal transition in planetary interiors. The point here is that the transition occurs at a pressure which depends on the chemical composition and the form of the EOS (Stevenson, 1998 for details on this problem).

Important applications of Bose gas are many in the theory of superconductivity. It is known in ordinary superconductors that pairs of charge carriers form a superfluid Bose condensate. The mechanism responsible for high temperature superconductivity has not yet been discovered (Marston, 1999), but it is almost certain that some form of a Bose gas will also occur. The net conclusion from this short list of examples of applicability testifies that studying the EOS formalism is far from being a mathematical "tour de force", but quite to the contrary, it paves the way to important astrophysical and physical applications.

References

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