Publ. Astron. Obs. Belgrade No. 60 (1998), 134 - 138 *Contributed paper*

# JACOBI INTEGRAL IN THE RESTRICTED CIRCULAR SCHWARZSCHILD-TYPE MANY-BODY PROBLEM

V. MIOC and M. STAVINSCHI
*Astronomical Institute of the Romanian Academy, Astronomical Observatory Bucharest*
*Str. Cuţitul de Argint 5, RO-75212 Bucharest, Romania*
*E-mail vmioc@roastro.astro.ro*
*E-mail magda@roastro.astro.ro*

**Abstract.** The planar Schwarzschild-type $(n+1)$-body problem with $n$ equal masses admits, for certain initial conditions, a class of exact solutions consisting of regular polygons (the $n$ equal masses at vertices, the $(n+1)$th mass at centre) of changing side length, and rotating nonuniformly around the centre. The various types of relative equilibria which are met among these configurations are taken as the basis for a 3-dimensional restricted problem. The existence of the Jacobi integral within this framework is proved.

## 1. INTRODUCTION

A special class of exact solutions in the Newtonian planar $(n+1)$-body problem $(n \geq 2)$ has been pointed out by Elmabsout (1988) in the case of $n$ equal masses initially placed at the vertices of a regular polygon centered in the $(n+1)$th mass. Geometrically, the solution represents a regular polygon of constant sides, uniformly rotating around the central mass. Elmabsout (1990, 1994, 1996) also investigated the stability of this configuration. Grebenicov (1997) found a new class of exact solutions for the above problem: the configuration of rotating regular polygon is preserved, but this time the side length and the angular velocity are changing.

Among the configurations pointed out by Grebenicov, there are stable relative equilibria (uniformly rotating polygon of constant dimensions). Taking such a configuration as the basis for a 3-dimensional restricted problem, Grebenicov (1998) proved that the respective problem admits the Jacobi integral; Gadomski (1998) extended this result to homogeneous potentials.

In the present paper we tackle the more general case of Schwarzschild-type fields (featured by potentials of the form $\alpha/r + \beta/r^3$; $r$ = distance between two particles; $\alpha$, $\beta$ = real nonzero constants), which model concrete problems belonging mainly to astronomy, but not only (see Stoica & Mioc (1997) and the references therein). Mioc *et al.* (1998) proved that, given the initial polygonal configuration considered by Elmabsout and Grebenicov, the respective $(n+1)$-body problem is equivalent to $n$ separate, identical, Schwarzschild-type two-body problems, therefore the regular polygonal configuration is kept all along the motion, but the side length and the rotational velocity are variable in general. The motion of every particle with respect to the central mass is governed by the solution of the Schwarzschild-type two-body problem, whose qualitative behaviour was fully described by Stoica & Mioc (1997), for the whole allowed interplay among field parameters, angular momentum, and total energy. Obviously, every kind of evolution in the two-body problem corresponds to a behaviour of the polygon in the $(n+1)$-body problem.

Among the possible evolutions of the polygon, there are many types of relative equilibria (polygons identical with the initial one, stable or unstable, rotating or fixed). Considering the motion of an infinitesimal mass in the Schwarzschild-type field generated by such an equilibrium configuration, we prove that the corresponding restricted problem admits the first integral of Jacobi.

## 2. POLYGONAL PROBLEM

Consider the $(n+1)$-body problem with masses $m_0$, $m_k = m \neq m_0$ $(k = \overline{1,n})$, let $\mathbf{q}_k = (\xi_k, \eta_k, \zeta_k) \in \mathbf{R}^3$, $k = \overline{0,n}$, be the position vectors in an inertial frame, and let $\mathbf{q} = (\mathbf{q}_0, \mathbf{q}_1, ..., \mathbf{q}_n) \in \mathbf{R}^{3n+3}$ be the configuration of the system. Let the $n+1$ particles be interacting according to a Schwarzschild-type law, characterized by the force function $U : \mathbf{R}^{3n+3} \setminus \Delta \to \mathbf{R}$, with

$$U(\mathbf{q}) = \sum_{0 \leq i < k \leq n} \left( \tilde{A}_{ki}/r_{ki} + \tilde{B}_{ki}/r_{ki}^3 \right). \tag{1}$$

Here $r_{ki} = \mid \mathbf{q}_k - \mathbf{q}_i \mid$, $\Delta = \cup_{0 \leq i < k \leq n} \{ \mathbf{q} \mid \mathbf{q}_i = \mathbf{q}_k \}$ stands for the collision set; $\tilde{A}_{ki}$, $\tilde{B}_{ki} : \mathbf{R}^2 \to \mathbf{R}$ feature the interaction between the $k$-th and the $i$-th particles: $\tilde{A}_{ki} = \tilde{A}(m_k, m_i) = \tilde{A}(m_i, m_k) = \tilde{A}_{ik}$ (of course, in our conditions, $\tilde{A}_{k0} = \tilde{A}_{j0} \neq \tilde{A}_{ji} = \tilde{A}_{ki}$, $k, i, j = \overline{1,n}$; similar relations hold for $\tilde{B}_{ki}$).

Consider the relative motion of the $n$ equal masses with respect to $m_0$ . The relative position vectors will be $\mathbf{r}_k = (x_k, y_k, z_k) = (\xi_k - \xi_0, \eta_k - \eta_0, \zeta_k - \zeta_0)$, hence $r_{ki} = \mid \mathbf{r}_k - \mathbf{r}_i \mid$. With the abridging notations (for $k, i = \overline{1,n}$):

$$\begin{bmatrix} A \\ B \end{bmatrix} := \frac{m_0 + m}{m_0 m} \begin{bmatrix} \tilde{A}_{k0} \\ \tilde{B}_{k0} \end{bmatrix}, \quad \begin{bmatrix} A' \\ B' \end{bmatrix} := \frac{1}{m} \begin{bmatrix} \tilde{A}_{ki} \\ \tilde{B}_{ki} \end{bmatrix}, \quad \begin{bmatrix} A'' \\ B'' \end{bmatrix} := \frac{1}{m_0} \begin{bmatrix} \tilde{A}_{k0} \\ \tilde{B}_{k0} \end{bmatrix}, \tag{2}$$

and $r_k = |\mathbf{r}_k|$, the relative motion equations can be written

$$\ddot{\mathbf{r}}_k = -\left( A/r_k^3 + 3B/r_k^5 \right) \mathbf{r}_k + \partial R_k(\mathbf{r}_k, t)/\partial \mathbf{r}_k, \quad k = \overline{1,n}. \tag{3}$$

where $R_k(\mathbf{r}_k, t) = \sum_{i=1, i \neq k}^{n} \left[ \left( A'/r_{ki} + B'/r_{ki}^3 \right) - \left( A''/r_i^3 + 3B''/r_i^5 \right) \mathbf{r}_k \cdot \mathbf{r}_i \right]$.

Obviously, the problem admits the ten well-known first integrals. Among the integration constants, we denote $\tilde{h}$ = energy constant, $\tilde{\mathbf{C}} (\in \mathbf{R}^3)$ = angular momentum constant.

Mioc *et al.* (1998) tackled the planar case ($z_k = 0$, $k = \overline{1,n}$, all along the motion) in polar coordinates $(r_k, \theta_k)$, and proved the following result:

THEOREM 1. *Let the masses $m_0$, $m_k = m \neq m_0$ $(k = \overline{1,n})$ be interacting according to a Schwarzschild-type law. Let $m_k$ be initially $(t = 0)$ situated at the vertices of* *a regular polygon centered in $m_0$, and let the initial velocities form a vector field symmetrical with respect to $m_0$. Then, all along the motion, the equal masses will form a regular polygon (centered in $m_0$), homothetic with the initial polygon, and rotating around $m_0$ with variable angular velocity. The motion of every mass $m$ with respect to $m_0$ is given by the solution of the Schwarzschild-type two-body problem.*

This means that the relative motion of every mass $m_k = m$, $k = \overline{1,n}$, is governed by the equations

$$\begin{cases} \ddot{r}_k - r_k\dot{\theta}_k^2 = -\alpha/r_k^2 - 3\beta/r_k^4, \\ r_k\ddot{\theta}_k + 2\dot{r}_k\dot{\theta}_k = 0, \end{cases} \tag{4}$$

with the first integrals of energy and angular momentum

$$\dot{r}_k^2 + r_k^2\dot{\theta}_k^2 - 2\alpha/r_k - 2\beta/r_k^3 = h;$$
$$r_k^2\dot{\theta}_k = C,$$

($h$ and $C$ being the respective constants), and with the regular polygonal solution

$$r_k(t) = r_i(t), \quad \theta_k(t) = \theta_1(t) + 2\pi(k-1)/n, \quad k, i = \overline{1,n}.$$

We have to emphasize that the parameters $\alpha$, $\beta$ (Mioc *et al.* 1997), $h$ and $C$, common for every pair ($m_0$, $m_k = m$), generally differ from the corresponding parameters featuring the initial $(n+1)$-body problem.

## 3. RELATIVE EQUILIBRIA

Studying the Schwarzschild-type two-body problem, Stoica & Mioc (1997) pointed out equilibrium configurations: circular motion ($C \neq 0$) or rest ($C = 0$) at relative distance $r_e$, stable ($r_e = r_{SE}$) or unstable ($r_e = r_{UE}$), with $r_{SE} = \left(-2\alpha - \sqrt{4\alpha^2 + 3hC^2}\right)/(3h)$, $r_{UE} = \left(-2\alpha + \sqrt{4\alpha^2 + 3hC^2}\right)/(3h)$ (for $h = 0$, $r_{UE} = r_{UE}^* := \sqrt{\beta/\alpha}$), provided the existence of the radical.

We are now in the position to state the following result:

THEOREM 2. *There are relative equilibrium solutions ($r_k = r_e$, $k = \overline{1,n}$) of the Schwarzschild-type polygonal $(n+1)$-body problem. In case $\tilde{C} \neq 0$, the regular polygon of constant size rotates uniformly with the angular velocity $\omega := \dot{\theta}_k = C/r_e^2$. In case $\tilde{C} = 0$ ($C = 0$), the polygon is fixed ($\omega = 0$).*

*Proof.* Taking into account the above mentioned results obtained by Stoica & Mioc (1997), as well as Theorem 1, the proof of Theorem 2 follows immediately.•

Let us mention the situations in which the interplay among $\alpha$, $\beta$, $h$, $C$ leads to relative equilibria in the Schwarzschild-type problem. These situations are: (a) $\alpha > 0$, $\beta > 0$, $C \neq 0$, $-4\alpha^2/(3C^2) < h$; for $h < 0$, there exist $r_{SE}$ and $r_{UE}$ (see above); for $h \leq 0$ and $C^2 = 4\sqrt{\alpha\beta}$, there exists $r_{UE}^*$; for $h > 0$, there exists $r_{UE}$; (b) $\alpha < 0$, $\beta > 0$, $h > 0$; for both $C \neq 0$ and $C = 0$, there exists $r_{UE}$; (c) $\alpha > 0$, $\beta < 0$, $-4\alpha^2/(3C^2) < h < 0$; for both $C \neq 0$ and $C = 0$, there exists $r_{SE}$.

## 4. ASSOCIATED RESTRICTED PROBLEM: JACOBI INTEGRAL

We shall take such a relative equilibrium configuration as the basis for a 3-dimensional restricted problem. So, consider the motion of an infinitesimal mass $\mu$ in the Schwarzschild-type field generated by the (rotating or fixed) constant size polygon. Let $\mathbf{d} = (x, y, z) \in \mathbf{R}^3$ and $\mathbf{d}_k = (x - x_k, y - y_k, z - z_k) \in \mathbf{R}^3$, $k = \overline{1, n}$, be respectively the position vectors of $\mu$ with respect to $m_0$ and $m_k (= m)$, and denote $\rho = |\mathbf{d}|$, $\rho_k = |\mathbf{d}_k|$. It is needless to say that the 3-dimensional frame in which we tackle the motion of $\mu$ is originated in $m_0$ and has the polygon plane as fundamental plane.

The relative motion equations of $\mu$ read

$$\ddot{\mathbf{d}} = -\left(\hat{A}/\rho^3 + 3\hat{B}/\rho^5\right)\mathbf{d} + \partial R(\mathbf{d}, t)/\partial \mathbf{d}, \tag{5}$$

where $R(\mathbf{d}, t) = \sum_{k=1}^{n}\left[\left(\hat{A}'/\rho_k + \hat{B}'/\rho_k^3\right) - \left(A''/r_k^3 + 3B''/r_i^5\right)\mathbf{d}\cdot\mathbf{d}_k\right]$, and we denoted $(\hat{A}, \hat{B}, \hat{A}', \hat{B}') = (A, B, A', B')(m = \mu)$ (see (2); we must emphasize that $\hat{A}, \hat{B}, \hat{A}', \hat{B}'$ are finite nonzero quantities).

THEOREM 3. *The restricted problem associated to the relative equilibrium solutions of the Schwarzschild-type polygonal* $(n+1)$*-body problem admits the Jacobi first integral.*

*Proof.* Let us pass to a uniformly rotating frame (in which $m_0$, $m_k = m$, $k = \overline{1, n}$, are fixed) via the transformations

$$\begin{cases} x = X\cos(\omega t) - Y\sin(\omega t), \\ y = X\sin(\omega t) + Y\cos(\omega t), \\ z = Z, \end{cases}$$

and similarly for $(x_k, y_k, 0) \longrightarrow (X_k, Y_k, 0)$, $k = \overline{1, n}$. In the new variables, (5) - written in scalar form - become

$$\begin{cases} \ddot{X} - 2\omega\dot{Y} - \omega^2 X = -\left(\hat{A}/\rho^3 + 3\hat{B}/\rho^5\right)X + \partial R^*/\partial X, \\ \ddot{Y} + 2\omega\dot{X} - \omega^2 Y = -\left(\hat{A}/\rho^3 + 3\hat{B}/\rho^5\right)Y + \partial R^*/\partial Y, \\ \ddot{Z} = -\left(\hat{A}/\rho^3 + 3\hat{B}/\rho^5\right)Z + \partial R^*/\partial Z, \end{cases} \tag{6}$$

where $R^*(X, Y, Z) = R(x, y, z, t)$. Obviously, $\rho^2 = X^2 + Y^2 + Z^2$ and $\rho_k^2 = (X - X_k)^2 + (Y - Y_k)^2 + Z^2$.

Multiplying respectively equations (6) by $\dot{X}$, $\dot{Y}$, $\dot{Z}$, adding the resulting expressions together, then integrating with respect to time, we get

$$\left(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2\right)/2 = \omega^2\left(X^2 + Y^2\right)/2 + \left(\hat{A}/\rho + \hat{B}/\rho^3\right) + R^* + h^*/2, \tag{7}$$

where $h^*$ is a constant of integration.

Observing that the relative equilibrium $\{r_k = r_e,\, \theta_k = \omega t + 2\pi(k-1)/n\}$ implies, in the new coordinates, $\sum_{k=1}^{n} X_k = 0$, $\sum_{k=1}^{n} Y_k = 0$, and replacing this in $R^*$, we easily obtain

$$\left(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2\right)/2 = \omega^2 \left(X^2 + Y^2\right)/2 + \left(\hat{A}/\rho\right) + \hat{B}/\rho^3 + \sum_{k=1}^{n} \left(\hat{A}'/\rho_k + \hat{B}'/\rho_k^3\right) + h^*/2,$$

which is nothing but the Jacobi integral. Theorem 3 is proved.

To end, we have to emphasize that, although the qualitative result is the same, the set of relative equilibria which form the basis of our restricted problem is more rich (as regards stability/instability, or rotation/rest) than those revealed by Grebenicov's (1998) or Gadomski's (1998) papers.

## References


Elmabsout, B.: 1988, *Celest. Mech. Dyn. Astron.* 41, 131.
Elmabsout, B.: 1990, *Celest. Mech. Dyn. Astron.* 49, 219.
Elmabsout, B.: 1994, *Dyn. Stabil. Syst.* 9, 305.
Elmabsout, B.: 1996, *Rom. Astron. J.* 6, 61.
Gadomski, L.J.: 1998, *Rom. Astron. J.* 8 (to appear).
Grebenicov, E.: 1997, *Rom. Astron. J.* 7 (to appear).
Grebenicov, E.: 1998, *Rom. Astron. J.* 8 (to appear).
Mioc, V., Stavinschi, M. & Orzaru, M.: 1998, *J. reine angew. Math.* (submitted).
Stoica, C., Mioc, V.: 1997, *Astrophys. Space Sci.* 249, 161.