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MHD WAVES IN CORONAL ARCADES

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Abstract. The MHD wave behavior in the solar corona with magnetic field having the shape of arcades is investigated. It is shown that a particular analytical solution to the linearized MHD equations can be obtained for perturbations with short wavelengths in the direction of the arcade tunnel.

Two possibilities are considered regarding the related wave frequency : the high frequency domain yields MHD waves propagating along the tunnel of the arcade as a fast MHD mode while the low frequencies produce two decoupled wave modes representing the Alfven and the slow magnetoacoustic wave, both modified by the gravity and the profile of the magnetic field. All these waves are stable, contrary to the case when the magnetic field is purely horizontal and when the magnetic buoyancy instabilities can set in.

1. STATIONARY MAGNETIC ARCADE

Magnetic fields having the shape of arcades are commonly found in the solar corona. They are usually considered as low plasma$-\beta$ fields meaning that the magnetic pressure significantly exceeds the thermal pressure of the ambient plasma. As the field lines of a coronal arcade emerge from the much denser photosphere, it is obvious that various photospheric processes will cause disturbances that can propagate further into the corona and be responsible for coronal heating and other phenomena (Čadež et.al 1995a, 1995b). It is, therefore, of particular interest to investigate the behavior of such perturbations which is being done either by numerical methods (Oliver et.al. 1993) or analytically (Čadež et.al.1994).

In this paper, we shall restrict our attention to the behavior of the so called *narrow perturbations* whose definition and properties are going to be given below.

Consider a magnetohydrostatic equilibrium of an ideal isothermal plasma with a magnetic arcade in a uniform gravity field along the vertical $z-$axis :

$$-v_s^2\nabla\rho_0 + \frac{1}{\mu_0}(\nabla \times \vec{B}_0) \times \vec{B}_0 + \rho_0\vec{g} = 0 \qquad (1)$$

where $v_s \equiv \sqrt{RT_0}$ is the isothermal ($\gamma = 1$) speed of sound.

The magnetic field of the arcade has two components, both lying in the vertical x, z-plane $\vec{B}_0 = (B_{0x}, 0, B_{0z})$, they are independent of the remaining horizontal $y-$coordinate and are given by the standard expressions :

$$B_{0x}(x,z) = B_{00}(\psi)\cos\left(\frac{x}{\lambda_B}\right)e^{-z/\lambda_B}, \quad B_{0z}(x,z) = -B_{00}(\psi)\sin\left(\frac{x}{\lambda_B}\right)e^{-z/\lambda_B}$$

The quantity $\psi(x, z)$ does not change along the field line and is known as flux function. If B_{00} is independent of ψ, the considered magnetic field becomes force-free and potential.

The field intensity $B_0 = B_{00}(\psi)\exp(-z/\lambda_B)$, shows an exponential decrease with the height z while the corresponding scale length λ_B remains unspecified at the moment.

For further calculations, it is convenient to replace the pair of Cartesian coordinates (x, z) by a new set θ and ψ in such a way that the new curvilinear coordinate lines $\psi = const.$ coincide with magnetic field lines while $\theta = const.$ lines are orthogonal to them. The y–coordinate remains unchanged and oriented horizontally, along the tunnel of the arcade. Thus :

$$\frac{\psi}{\lambda_B} = \cos\left(\frac{L}{\lambda_B}\right) - \cos\left(\frac{x}{\lambda_B}\right)e^{-z/\lambda_B}, \qquad \frac{\theta}{\lambda_B} = \sin\left(\frac{x}{\lambda_B}\right)e^{-z/\lambda_B}, \qquad y = y$$

The quantity L is used only to define the referent $\psi = 0$ coordinate line and its value can be chosen arbitrarily.

The standard vector field operators take now the following form :

$$\nabla p = \frac{1}{h}\frac{\partial p}{\partial \psi}\hat{e}_\psi + \frac{1}{h}\frac{\partial p}{\partial \theta}\hat{e}_\theta + \frac{\partial p}{\partial y}\hat{e}_y \quad \text{and} \nabla\cdot\vec{v} = \frac{1}{h^2}\left[\frac{\partial}{\partial\psi}(hv_\psi) + \frac{\partial}{\partial\theta}(hv_\theta)\right] + \frac{\partial v_y}{\partial y}$$

for the gradient of a scalar p and for the divergence of a vector $\vec{v}$ respectively, and :

$$\nabla\times\vec{v} = \frac{1}{h^2}\begin{vmatrix} h\hat{e}_\psi & h\hat{e}_\theta & \hat{e}_y \\ \dfrac{\partial}{\partial\psi} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial y} \\ hv_\psi & hv_\theta & v_y \end{vmatrix} \tag{2}$$

for the curl of a vector $\vec{v}$. Here $h \equiv e^{z/\lambda_B} = \left[\left(\frac{\theta}{\lambda_B}\right)^2 + \left(\cos\frac{L}{\lambda_B} - \frac{\psi}{\lambda_B}\right)^2\right]^{-1/2}$

The hydrostatic balance equation (1) can now be expressed in components which gives :

$$v_s^2\frac{\partial}{\partial\theta}\ln(\rho_0 h^\delta) = 0 \quad \text{and} \quad v_s^2\frac{\partial}{\partial\psi}\ln(\rho_0 h^\delta) + \frac{B_{00}^2}{\mu_0\rho_0 h^2}\frac{d}{d\psi}\ln B_{00} = 0 \tag{3}$$

where $\delta \equiv g\lambda_B/v_s^2$.

According to the first equation in Eqs.(3), the expression $\rho_0 h^\delta$ depends only on the variable ψ while the second equation indicates the same property for $\rho_0 h^2$. Therefore $\delta = 2$, meaning that $\lambda_B = 2\lambda$ with $\lambda \equiv v_s^2/g$ being the standard scale height of an isothermal and non magnetized atmosphere. However, in cases when $B_{00} = const.$ (a potential magnetic field) and/or $B_{00}^2/(\mu_0\rho_0 h^2) \gg v_s^2$ (the low plasma β medium), the value of the parameter δ remains arbitrary. In both cases, namely, the spatial distribution of the plasma density does not affect the magnetic field distribution, they are mutually independent and $\delta = \lambda_B/\lambda$ may have any value.

2. LINEARIZED EQUATIONS

To investigate the behavior of small amplitude isothermal perturbations of the described equilibrium state, we start from the standard set of linearized MHD equations :

$$\frac{\partial \rho_1}{\partial t} + \nabla \cdot (\rho_0 \vec{v}) = 0, \qquad \frac{\partial \vec{B}_1}{\partial t} = \nabla \times (\vec{v} \times \vec{B}_0), \qquad p_1 = v_s^2 \rho_1,$$

$$\rho_0 \frac{\partial \vec{v}}{\partial t} = -\nabla p_1 + \frac{1}{\mu_0}(\nabla \times \vec{B}_0) \times \vec{B}_1 + \frac{1}{\mu_0}(\nabla \times \vec{B}_1) \times \vec{B}_0 + \rho_1 \vec{g} \tag{4}$$

Since the basic state is stationary and does not depend on the variable y, any of the perturbed quantities can be taken as a product of a harmonic function of both the time variable t and the spatial coordinate y, i.e. $exp(-i\omega t + ik_y y)$, and a $(\psi, \theta)-$dependent amplitude. In this case the Eqs.(4) reduce to the following system of equations for the perturbation amplitudes :

$$h^2\omega^2 V_\psi + v_A^2 \frac{\partial^2 V_\psi}{\partial \theta^2} + i\omega v_s^2 \left(\frac{\partial \Pi}{\partial \psi} - \frac{v_A^2}{2v_s^2 + v_A^2} \frac{d}{d\psi} \ln v_A^2 \, \Pi \right) = 0 \tag{5}$$

$$h^2\omega^2 V_\theta + v_T^2 \frac{\partial^2 V_\theta}{\partial \theta^2} + i\omega \frac{c^4}{v_s^2 + v_A^2} \frac{\partial \Pi}{\partial \theta} = 0 \tag{6}$$

$$h^2\omega^2 V_y + v_A^2 \frac{\partial}{\partial \theta} \left[\frac{1}{h} \frac{\partial}{\partial \theta} (h V_y) \right] - \omega h v_s^2 k_y \Pi = 0 \tag{7}$$

$$i\omega \Pi = \frac{v_s^2 + v_A^2}{v_s^2} \left(\frac{\partial V_\psi}{\partial \psi} + ik_y h V_y \right) + \frac{\partial V_\theta}{\partial \theta}. \tag{8}$$

Here $v_T^2 \equiv v_s^2 v_A^2/(v_s^2 + v_A^2)$, $\vec{V} \equiv \vec{v}/h$ and Π is the total pressure perturbation normalized to the local gas pressure of the unperturbed medium, i.e. $\Pi \equiv p_{tot}/(v_s^2 \rho_0)$ and $p_{tot} \equiv v_s^2 \rho_1 + B_{1\theta} B_{00}/(h\mu_0)$.

The obtained set of Eqs.(5)-(8) reduces to the result obtained by (Goedbloed, 1971) for the 2D perturbations with $k_y = 0$ when the Alfven mode, propagating along the tunnel of the arcade, was decoupled from the two remaining modes propagating in the cross sectional plane of the arcade.

In our case, now, all three modes are coupled and an overall analysis of their behavior, based upon the solution of Eqs.(5)-(8), is rather complicated. However, it is possible to examine a particular domain of these perturbations when they have very short wavelengths along the $y-$direction, i.e. the case when k_y takes comparatively large values, contrary to the $k_y = 0$ considered earlier (Goedbloed, 1971).

3. NARROW PERTURBATIONS

The perturbations with large k_y are of a narrow shape in the lateral $y-$direction and will be referred to as *narrow perturbations*. In other words, we shall consider narrow perturbations as those disturbances whose $y-$derivatives are much larger than either their $\psi-$ and $\theta-$derivatives or the corresponding derivatives of the basic state quantities. Symbolically :

$$\left|\frac{\partial}{\partial y}\right| = k_y \gg \left|\frac{\partial}{\partial \psi}\right|, \left|\frac{\partial}{\partial \theta}\right| \tag{9}$$

As to the time derivatives, i.e. the perturbation frequency ω, there are two possibilities : the low frequency and the high frequency domains when the $y-$derivatives are much larger and of the same order respectively as compared to the related time derivatives. Symbolically this would be :

$$\left|\frac{\partial}{\partial y}\right| = k_y \gg \frac{1}{v_s + v_A}\left|\frac{\partial}{\partial t}\right| = \frac{\omega}{v_s + v_A} \text{ the low frequency case} \tag{10}$$

$$\left|\frac{\partial}{\partial y}\right| = k_y \sim \frac{1}{v_s + v_A}\left|\frac{\partial}{\partial t}\right| = \frac{\omega}{v_s + v_A} \text{ the high frequency case} \tag{11}$$

To see what this means, one can consider the typical coronal conditions when $v_A \geq v_s$ and $v_A \sim 10^4 km/s$. In this case the low frequency condition (10) becomes

$$k_y \equiv \frac{2\pi}{\lambda_y} \gg \frac{\omega}{v_s + v_A} > \frac{\omega}{v_A} \equiv \frac{2\pi}{\tau v_A}$$

Consequently, the conditions (10) and (11) now relate the perturbation lateral extend λ_y to the oscillation time period τ as follows :

$$\frac{\lambda_y}{\tau} \ll v_A \sim 10^4 km/s \quad \text{and} \quad \frac{\lambda_y}{\tau} \sim v_A \sim 10^4 km/s$$

for the low frequency and the high frequency cases, respectively.

The low frequency solutions. Consider the low frequency case first with orderings given by (9) and (10). As the large wave number k_y enters only the equations (7) and (8), they can be expressed as follows :

$$\omega\Pi = \frac{1}{hv_s^2 k_y}\left\{h^2\omega^2 V_y + v_A^2 \frac{\partial}{\partial \theta}\left[\frac{1}{h}\frac{\partial}{\partial \theta}(hV_y)\right]\right\} \to 0$$

$$V_y = \frac{1}{ik_y h}\left[\frac{v_s^2}{v_s^2 + v_A^2}\left(i\omega\Pi - \frac{\partial V_\theta}{\partial \theta}\right) - \frac{\partial V_\psi}{\partial \psi}\right] \to 0$$

The Eqs.(5)-(8) finally take a simple form :

$$\frac{\partial^2 V_\psi}{\partial \theta^2} + h^2(\psi,\theta)\frac{\omega^2}{v_A^2(\psi)}V_\psi = 0, \qquad \frac{\partial^2 V_\theta}{\partial \theta^2} + h^2(\psi,\theta)\frac{\omega^2}{v_T^2(\psi)}V_\theta = 0, \tag{12}$$

$$\Pi = 0 \qquad \text{and} \qquad V_y = 0 \tag{13}$$

where $h(\psi, \theta)$ is given in (2).

According to (13), the considered low frequency narrow perturbations are in a total pressure equilibrium and cause no fluid motions in the y−direction. In addition, it can be easily shown from the condition $\nabla \cdot \vec{B}_1 = 0$, that the perturbed magnetic field has no y−component either, i.e. that $B_{1y} = 0$ in this case.

The equations (12) are now decoupled and contain no ψ−derivatives. The initial system of four coupled partial differential equations is thus reduced to only two mutually independent ordinary differential equations. The variable ψ can be treated as a parameter and, consequently, the solutions can be obtained for arbitrary Alfven speed distributions $v_A^2(\psi)$ by integrating Eqs.(12) over the variable θ only.

Both Eqs.(12) are of the same form and can be expressed as :

$$\frac{d^2 V_i}{du^2} + \frac{a_i^2}{1+u^2} V_i = 0, \qquad \text{here} \qquad i = \psi, \theta$$

where :

$$u = \frac{\theta}{\lambda_B \left| \cos\left(\frac{L}{\lambda_B}\right) - \frac{\psi}{\lambda_B} \right|} = \tan\left(\frac{x}{\lambda_B}\right),$$

$$a_\psi^2 = \frac{\omega^2 \lambda_B^2}{v_A^2(\psi)} \quad \text{and} \quad a_\theta^2 = \frac{\omega^2 \lambda_B^2}{v_T^2(\psi)} = \frac{v_s^2 + v_A^2(\psi)}{v_s^2 v_A^2(\psi)} \omega^2 \lambda_B^2 \tag{14}$$

The approximate solution of the above equation can be obtained by means of the WKB method in the following form :

$$V_i = C^{(+)} V_i^{(+)} + C^{(-)} V_i^{(-)}, \qquad C^{(+)}, C^{(-)} = const.$$

where the two linearly independent solutions, $V_i^{(+)}$ and $V_i^{(-)}$, are given by :

$$V_i^{(+)} = a_i^{-1/2}(1+u^2)^{1/4} \cos\left\{ a_i \ln\left[(1+u^2)^{1/2} + u\right] + a_i \ln 2 \right\},$$

$$V_i^{(-)} = a_i^{-1/2}(1+u^2)^{1/4} \sin\left\{ a_i \ln\left[(1+u^2)^{1/2} + u\right] + a_i \ln 2 \right\}.$$

The above solution is valid provided the condition for the WKB approximation is satisfied which, in this case, means that the following inequality holds :

$$\frac{|2 - u^2|}{4a_i^2(1+u^2)} \ll 1 \tag{15}$$

It can be easily shown that (15) is satisfied for any value of u if the coefficient a_i is large enough : $a_i \gg 1/\sqrt{2}$. Since $a_\theta \geq a_\psi$ according to definitions (14), this condition becomes :

$$a_\psi \equiv \frac{2\pi\lambda_B}{\tau v_A} \gg \frac{1}{\sqrt{2}} \tag{16}$$

where $\tau \equiv 2\pi/\omega$ is the oscillation time period.

To estimate the condition (16) let us introduce some real values for the solar corona by taking $\lambda_B = 120,000\,km$ and $v_A = 10^4 km/s$. In this case (16) requires $\tau \ll 24\pi\sqrt{2} \approx 110s$. or that oscillations with time period up to 1.5 min. can be treated by the considered WKB method.

The high frequency solutions. In the case when the frequency ω is taken sufficiently large to make the terms with ω^2 of the same order of magnitude as those with k_y^2, the Eqs.(5)-(8) reduce to a simple set of algebraic equations again :

$$h^2\omega^2 V_y - hv_s^2 k_y \omega \Pi = 0, \quad \frac{v_s^2 + v_A^2}{v_s^2} hk_y V_y - \omega\Pi = 0 \text{ and } V_\psi = V_\theta = 0 \qquad (17)$$

As can be seen, the high frequency narrow perturbations induce fluid motions in the $y-$direction only and also a varying total pressure. The resulting wave propagates in the $y-$direction, according to the dispersion equation

$$\frac{\omega^2}{v_s^2 + v_A^2(\psi)} - k_y^2 \doteq 0$$

that follows from Eq.(17). This is the fast MHD mode with its phase velocity depending on the variable ψ.

4. CONCLUSION

As can be seen from both solutions obtained in the low and in the high frequency domain, the considered magnetic field configuration is stable with respect to the narrow perturbations which is not the case when the field is purely horizontal (Gilman, 1970). The stability of a magnetic arcade is due to the addit˙onal magnetic field curvature stress that opposes the destabilizing action of the magnetic buoyancy.

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