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ON NEARLY CIRCULAR ORBITS

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Abstract. Nearly circular orbits are characteristic for thin discs of spiral galaxies, the subsystem and the type of galaxies to which the Sun and the Milky Way belong. Such orbits are studied in a way different from the usual one (epicycles). Formulae wherein the total energy and the conserved angular momentum component are expressed in terms of the mean distance and eccentricity are derived. The classical results concerning the sinusoidal dependence of distance on time and the ratio of the circular period to the anomalistic one for the same mean distance are confirmed.However, it is shown that, whereas the dependence on time even for a very low eccentricity begins to deviate from the sinusoidal form noticeably, the period ratio remains practically unaffected for eccentricities almost as high as 0.5.

1. INTRODUCTION

Orbits of stars which belong to the thin discs of spiral galaxies are known to be almost circular. This motion can be decomposed into the motions with respect to the axis of symmetry and with respect to the midplane. The former one is then analogous to the case of spherical symmetry (central motion) because of the conservation of one component (along axis of symmetry) of the angular momentum. Due to the assumption of low eccentricity it has been reduced to the linear harmonic oscillations (epicyclic approximation). On this subject information can be found in many books (e.g. Binney and Tremaine 2008, 3.2.3, p. 162).

More detailed studies of the motion of stars belonging to the thin disc in the solar neighbourhood show that the semi-separability assumption (see next section) is more realistic than the separability one, i.e. the maximum distance from the midplane is affected by the distance to the axis of symmetry, as well as that the dependence of the coordinates on time does not strictly follow simple sinusoidal forms (e.g. Stojanović 2015). In the same paper it is also inferred that, if eccentricity is low enough, assuming a power law dependence of the circular speed on the distance realistic orbital elements are obtained.

This is a short presentation only. The full account is to be published elsewhere.

2. BACKGROUND

The motion of a test particle is studied in an inertial reference frame. The coordinate system $Oxyz$ with origin at the centre of an axially symmetric stellar system (galaxy) is right-handed. Besides, the potential governing the motion is stationary. Because of the axial symmetry it is convenient to replace the rectangular coordinates x and y with the cylindrical ones R and θ ($x = R\cos\theta$, $y = R\sin\theta$). The motion of the test particle is described by means of the Lagrange equations:

$$\ddot{R} - \frac{J_z^2}{R^3} = \frac{\partial \Pi}{\partial R} \ ; \tag{1}$$

$$\ddot{z} = \frac{\partial \Pi}{\partial z} \ .$$

In these two equations the potential is denoted as Π ($\Pi \geq 0$) and J_z is the designation for the component of the specific (per unit mass) angular momentum which is conserved.

The next assumption is that for the test particle motion it is always valid $z \approx 0$. In such conditions the potential $\Pi(R, |z|)$ is approximately equal to the corresponding potential in the plane $z = 0$ (midplane), $\Pi(R, 0)$ (semi-separability). Then the potential in (1) will be $\Pi(R, 0)$, which is independent of z. As a consequence, the partial derivative may be replaced by the total one and since the state is steady, the following quasi-integral of motion will be valid

$$E_p \approx \text{const}, \quad E_p = \frac{1}{2}V_p^2 - \Pi(R, 0);$$

$$V_p = \sqrt{\dot{R}^2 + \frac{J_z^2}{R^2}} \ .$$

For convenience the two integrals of motion, E_p and J_z, will be here replaced by the mean distance and eccentricity. Both will be defined via the pericentric and apocentric distances, R_p and R_a, respectively. The mean distance is

$$R_m = \frac{R_a + R_p}{2} \ ;$$

and eccentricity e

$$e = \frac{R_a - R_p}{R_a + R_p} \ .$$

In the case of the pure circular motion it will be: $e = 0$ and $R = R_m$ =const. A nearly circular motion ipso facto would be $e \approx 0$ and $R \approx R_m$. By introducing variation of the distance δR, $\delta R = R - R_m$, and the effective potential (subtracting square of tangential velocity component from potential), bearing in mind that the variation is expected to be small equation (1) can be reduced to the form of harmonic oscillations. However, the prerequisite is that in the binomial on e contained in J_z^2 the linear term should be absent. This prerequisite has not been examined. Therefore, in what follows the two integrals E_p and J_z will be expressed in terms of the mean distance and eccentricity.

The energy is expressed in the following way

$$E_p = \frac{1}{2} f u_c^2(R_m) - \Pi(R_m) \ ,$$

where f is dimensionless and u_c is the circular speed. As for J_z, it is borne in mind that at both $R = R_p$ and $R = R_a$ it is valid $\dot{R} = 0$. The energy conservation is used so that the potential at the extremal distances is expanded in series where the terms higher than two are neglected. The quantity J_z^2 is represented twice, once as $J_z^2(R_p)$ and once as $J_z^2(R_a)$. Since they must be equal to each other, one finds the following solutions

$$f = 1 \; ; \tag{2}$$

$$J_z^2 = R_m^2 u_c^2(R_m)[1 - (3 - \gamma_2)e^2] \; . \tag{3}$$

The dimensionless quantity γ_2, more precisely $\gamma_2(R_m)$, is defined via the second derivative of the potential at $R = R_m$ in this way

$$\frac{d^2\Pi(R,0)}{dR^2}(R_m) = \gamma_2(R_m)\frac{u_c^2(R_m)}{R_m^2} \; .$$

However, in addition to (2) and (3) there is another solution. It is based on the fact that the effective potential has the same value at $R = R_p$ and at $R = R_a$. Then one finds that J_z^2 is proportional to $[R_m u_c(R_m)]^2$ multiplied by $(1 - e^2)^2 \approx 1 - 2e^2$. The solutions are

$$f = 1 + [1 - \gamma_2(R_m)]e^2 \; ;$$

$$J_z^2 = R_m^2 u_c^2(R_m)[1 - 2e^2] \; .$$

In order to understand the meaning of these solutions another approximation is introduced, namely that in the surroundings of R_m the behaviour of the circular speed obeys this law

$$u_c(R) \propto R^\delta \; ; -\frac{1}{2} \leq \delta \leq 1$$

Since this is a local approximation only, δ will be in fact $\delta(R_m)$. It is easy to establish a relation between $\gamma_2(R_m)$ and $\delta(R_m)$. Thus one has

$$\delta(R_m) \leq 0, f = 1 \; , J_z^2 = R_m^2 u_c^2(R_m)[1 - 2(1 + \delta)e^2] \; .$$

$$\delta(R_m) \geq 0, f = 1 + 2\delta e^2 \; , J_z^2 = R_m^2 u_c^2(R_m)(1 - 2e^2) \; .$$

It is easy to see that the two solutions are congruent if $\delta(R_m) = 0$. For convenience (R_m) after δ is omitted in the formulae.

In order to obtain the dependence $R(t)$, provided that $e \approx 0$, the classical procedure is used. In other words, since $\dot{R} \equiv dR/dt$, one obtains an integral in R with a function of R in the denominator, then the approximations following from the condition $e \approx 0$ are taken into account and, finally, it is obtained

$$R = R_m \left[1 + e \sin\left(\frac{2\pi}{P_a}t - \frac{\pi}{2}\right)\right] \; . \tag{4}$$

Here P_a is the anomalistic period; for $e \approx 0$ it is given as

$$P_a(R_m) = P_{circ}(R_m)\left[2\left(\delta(R_m)+1\right)\right]^{-\frac{1}{2}} \;, P_{circ}(R_m) = \frac{2\pi R_m}{u_c(R_m)} \; . \tag{5}$$

These results are well known, but when they are obtained following the procedure applied here, then the term in the J_z^2 expression containing e^2 must be taken into account. If it is omitted, in the denominator of the integral we would have a square root from a negative value. In addition, it is seen that in the binomial in e for J_z^2 the linear term is zero, the prerequisite to obtain linear harmonic oscillations in the Lagrange equation.

3. DISCUSSION AND CONCLUSIONS

The behaviour of (4) and (5) will be considered when the eccentricity value is varied. For a test particle (star) satisfying the condition $e \approx 0$ the motion in R is examined numerically for a realistic potential. The general conclusion is that the period relation (5) suffers only minor changes when eccentricity increases, even to as high values as $e = 0.5$. On the other hand, time dependence (4) with increasing eccentricity rather rapidly shows changes, in particular that $R = R_m$ is attained, not after the quarter of period as it follows from (5), but earlier. In other words, if it is assumed $R = R_p$ corresponds to $t = 0$, then the time t_m when $R = R_m$ is shorter than $(1/2)P_a - t_m$. The test particle stays at distances exceeding R_m longer than at those shorter than R_m. As an improvement the following relation may be proposed

$$\frac{R}{R_m} = 1 + e\sin\left(\frac{2\pi}{P_a}t - \frac{\pi}{2}\right) + \frac{e^2}{\sqrt{2(1+\delta)}}\cos^2\left(\frac{2\pi}{P_a}t - \frac{\pi}{2}\right) \; .$$

Of course, it is $\delta = \delta(R_m)$. This relation is proven more realistic because it yields the time interval between $t = t_m$ and $t = (1/2)P_a$ longer than that between $t = 0$ and $t = t_m$ and their ratio increases with increasing eccentricity.

References

Binney, J., Tremaine, S.: 2008, Galactic Dynamics, Second Edition, Princeton University Press.

Stojanović, M.: *Serbian Astronomical Journal*, **191**, pp. 75-80