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ALGEBRAIC DEPENDENCIES AND REPRESENTATIONS OF COSMOLOGICAL PARAMETERS

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Abstract. Cosmological parameters are taken usually as functions of time. These dependencies are transcendental in general and cannot be represented in the closed form by elementary functions. We show that there are algebraic dependencies between the main cosmological parameters: the scale factor, Hubble parameter, redshift and density parameters related to the cosmological constant, space curvature, radiation and baryonic matter. We exhibit and discuss these dependencies and give several examples of their use in computing certain events in the evolution of the universe.

1. INTRODUCTION

Accurate determination of cosmological parameters is an important task in the study of ΛCDM model, as they enable computing of the main events in the past and prediction of the future evolution of the universe.

Cosmological parameters are taken usually as functions of time t. These dependencies are transcendental in general and cannot be represented in the closed form by elementary functions due to the elliptic integrals which come out as the equivalent solutions of Friedmann equations:

$$t = \frac{1}{H_t}\int_0^1 \frac{s\,ds}{\sqrt{\Omega_{rt} + \Omega_{mt}s + \Omega_{kt}s^2 + \Omega_{\Lambda t}s^4}}, \quad \text{or} \tag{1}$$

$$t \equiv \mathcal{J}(a) = \frac{1}{H_0}\int_0^a \frac{s\,ds}{\sqrt{\Omega_{r0} + \Omega_{m0}s + \Omega_{k0}s^2 + \Omega_{\Lambda 0}s^4}}. \tag{2}$$

Parameters appearing in these integrals are the scale factor a (taken as an independent variable in (2)), Hubble parameter $H_t = H(t)$ and density parameters $\Omega_{it} = \Omega_i(t)$ at time t related to the cosmological constant Λ, space curvature κ, radiation r and baryonic matter m. We take $H_0 = H(t_0)$, the value of the Hubble parameter at the time moment t_0. The similar convention is applied to the other parameters.

We remind that Friedmann equations, discovered by Alexander Friedmann in 1922, are usually stated as a system consisting of the first and second order differential equations:

$$\begin{aligned} &\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3}, && \text{Friedmann equation,} \\ &\frac{\ddot a}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda c^2}{3}, && \text{Acceleration equation,} \\ &\dot\rho + 3\frac{\dot a}{a}\left(\rho + \frac{p}{c^2}\right) = 0, && \text{Fluid equation.} \end{aligned} \tag{3}$$

These equations represent the core of the ΛCDM model. The function $a(t)$ is the expansion scale factor and it describes the evolution of our universe. Other parameters appearing in Friedmann equations are the pressure $p(t)$ and the energy density $\rho(t)$. Both integrals (1) and (2) can be derived, for example, from the well-known identities

$$H^2/H_0^2 = \Omega_{\Lambda 0} + \Omega_{k0}a^{-2} + \Omega_{m0}a^{-3} + \Omega_{r0}a^{-4}, \quad p = \frac{1}{3}c^2\rho_r. \tag{4}$$

We note that $\mathcal{J}(a)$ is the inverse function of the scale factor $a(t)$, i.e. for any $b > 0$

$$\mathcal{J}(b) = t \text{ if and only if } a(t) = b.$$

Hence, (2) is a parametrization of the cosmic time t in respect to the scale factor a.

2. ALGEBRAIC DEPENDENCIES

We show that there are algebraic dependencies between the main cosmological parameters. Besides discussion of these dependencies, we also exhibit them including those corresponding to the contemporary measurements. Our main motivation was to apply these dependencies in the study of the evolution of the universe.

Two possible applications of found dependencies are presented. The first one is the representation of cosmological parameters in respect to some chosen parameter. The second one is the computation of values of all cosmological parameters at particular points in the evolution of the universe, such as the transition point from radiation dominated to matter dominated era, recombination, or the starting point of the accelerated expansion of the universe.

In order to state precisely these dependencies, we first fix notation. In general relativity first Friedmann equation is

$$H(t)^2 \equiv \left(\frac{\dot R(t)}{R(t)}\right)^2 = \frac{8\pi G}{3}\rho(t) - \frac{kc^2}{R(t)^2} + \frac{\Lambda c^2}{3}, \tag{5}$$

where $R = R(t)$ is the curvature radius at time t and $k = 0, 1, -1$ is the curvature index in the FLRW metric. Also, $\rho(t)$ is the mass density which includes rest mass energy and other forms of energy (e.g., energy of photons). Furthermore

$$a(t) = R(t)/R(t_0), \quad H(t) = \dot R(t)/R(t) = \dot a(t)/a(t), \tag{6}$$

where t_0 is the present time, $H = H(t)$ is the Hubble parameter and $a = a(t)$ is the scale factor, the normalization of $R(t)$.

Let t_0 and τ be two time moments. We take that t_0 is a constant and may stand for the present time. In fact t_0 is a part of the boundary conditions for Friedmann equations.

Let us write $R_0 = R(t_0)$, $R_\tau = R(\tau)$, $H_0 = H(t_0)$, $H_\tau = H(\tau)$, $\Omega_{i\tau} = \Omega_i(\tau)$ and $\Omega_{i0} = \Omega_i(t_0)$. The redshift is denoted by z. Then we have the following algebraic relations between the cosmological parameters.

$$\begin{aligned} a_\tau &= 1/x, \qquad R_\tau = R_0/x, \qquad H_\tau = H_0/\sqrt{y}, \qquad z = x - 1, \\ \Omega_{\Lambda\tau} &= \Omega_{\Lambda 0} y, \quad \Omega_{k\tau} = \Omega_{k0} x^2 y, \quad \Omega_{m\tau} = \Omega_{m0} x^3 y, \quad \Omega_{r\tau} = \Omega_{r0} x^4 y, \\ y &= \frac{1}{\Omega_{\Lambda 0} + \Omega_{k0} x^2 + \Omega_{m0} x^3 + \Omega_{r0} x^4}. \end{aligned} \tag{7}$$

The algebraic dependencies (7) are determined using the standard formulas found in the literature which relate cosmological parameters, for example the defining formulas for density parameters:

$$\begin{aligned} \Omega_\Lambda(t) &= \frac{\Lambda c^2}{3H(t)^2}, \quad \text{cosmological constant density}, \\ \Omega_k(t) &= -\frac{kc^2}{R(t)^2 H(t)^2} = -\frac{\kappa_0 c^2}{a(t)^2 H(t)^2}, \quad \text{curvature density}, \\ \Omega(t) &= \frac{8\pi G}{3H(t)^2}\rho(t), \quad \text{mass density}, \\ \Omega_m(t) &= \frac{8\pi G}{3H(t)^2}\rho_m(t), \quad \text{rest mass density}, \\ \Omega_r(t) &= \frac{8\pi G}{3H(t)^2}\rho_r(t), \quad \text{radiation density}. \end{aligned} \tag{8}$$

In (7) R_0, H_0, Ω_{i0} are constants and they form the boundary condition for Friedmann equations. We may think of them as of the measured values at t_0, while $a_\tau, R_\tau, H_\tau, \Omega_{i\tau}$ are unknowns and x, y are auxiliary variables.

The relations in (7) give an algebraic parameterizations of basic cosmological parameters in respect to x. Obviously, the parameter y is immediately eliminable from this system replacing all appearances of y using the last formula in (7), so the dependencies (7) consist of eight equations and nine unknowns, eight of them representing cosmological parameters. Also, we see that instead of the parameter x we could take the redshift z. But it is also obvious that after these substitutions we would loose the simplicity and easy readability of the algebraic dependencies represented by (7). In fact, instead of x we could take in this system as a free parameter any time-like parameter, i.e. that one which is monotonous in time, either increasing, or decreasing, for example the scale factor a. But all such modifications lead to more complicated set of formulas than (7).

3. APPLICATIONS

The usual approach found in the literature in computing some events in the evolution of the universe, is to locate the epoch at which the event most certainly appear and then to find an approximation valid for this epoch

$$\tau = G(a, H, \Omega_i) \tag{9}$$

of the solution (1), or its alternative (2). Then using equation (9), time t_0 and the related values of cosmological parameters of the event are computed.

An example of the standard approach is the derivation of Carroll-Press-Turner formula (1992) for pressureless flat universe with cosmological constant. These assumptions correspond to a universe with the curvature $\kappa = 0$ for the period following radiation era, i.e. approximately $\Omega_{r0} = 0$ and $\Omega_{k0} = 0$. Hence

$$\begin{aligned} I &= \int_0^1 \frac{s}{\sqrt{\Omega_{m0} s + \Omega_{\Lambda 0} s^4}}\, ds \\ &= \frac{-\ln(\Omega_{m0}\Omega_{\Lambda 0}) + 2\ln(\Omega_{\Lambda 0} + \sqrt{\Omega_{\Lambda 0}}\sqrt{\Omega_{m0} + \Omega_{\Lambda 0}})}{3\sqrt{\Omega_{\Lambda 0}}}. \end{aligned}$$

Taking $\Omega_{m0} = \Omega_0$ and by $\Sigma_i \Omega_i = 1$, we have $\Omega_0 + \Omega_{\Lambda 0} = 1$, and after some simplifications we obtain the desired formula:

$$H_0 t_0 = \frac{2}{3} \frac{1}{\sqrt{1 - \Omega_0}} \ln\left(\frac{1 + \sqrt{1 - \Omega_0}}{\sqrt{\Omega_0}}\right).$$

Now we give some simple computational examples based on the algebraic dependencies (7). The idea is to find an extra relation which connects the cosmological parameters. With this supplement the system (7) will have nine equations with nine unknowns, what would lead to a solution of the system. For t_0 we take the present time and for values of the parameters at t_0 we take a set of mean currently measured values (Particle Data Group, http://pdg.lbl.gov):

$$\begin{gathered} a_0 = 1, \quad H_0 = 67.4\,(\mathrm{km/s})/\mathrm{Mpc} = 2.1843 \cdot 10^{-18} s^{-1}, \\ \Omega_{\Lambda 0} = 0.685, \quad \Omega_{k0} = 0.0007, \quad \Omega_{m0} = 0.3164, \quad \Omega_{r0} = 0.0000538. \end{gathered}$$

We also take $\Omega_m = \Omega_b + \Omega_c + \Omega_\nu$, where Ω_b, Ω_c and Ω_ν are densities respectively of baryonic mass, cold dark matter and neutrinos.

Before we proceed to examples, we note that the presented method gives theoretically the most accurate calculations of cosmological events. The main part of the method is the numerical computation of the elliptic integral which represents the exact analytical solution (2) of Friedmann equations. No approximation of formulas are assumed as it is usually done in the literature. Hence, the influence of even small values of certain parameters are taken into account which are usually neglected in computations for a chosen cosmological epoch. The other part of computation comes from algebraic dependencies (7) which represent the usual physical laws. Hence, the accuracy of the method, if ΛCDM model is assumed, depends only on the the numerical methods for computing integrals which accuracies are arbitrary high today and the accuracy of the measurements of the contemporary cosmological data which are taken as the input values for the computation.

Age of universe. In this example we take $\tau = t_0$, i.e. we compute the age of the universe. Hence $x = 1$ and we compute

$$t_0 = \frac{1}{H_0} \int_0^1 \frac{s\, ds}{\sqrt{\Omega_{r0} + \Omega_{m0} s + \Omega_{k0} s^2 + \Omega_{\Lambda 0} s^4}} = 13.7815\,\mathrm{Gyr}. \tag{10}$$

Double expansion. In this example we compute time τ when the universe will double its expansion, i.e. $a_\tau = 2$. Then $x = 0.5$, so using (7) we obtain

$$\begin{gathered} y_\tau = 1.3828, \quad H_\tau = 1.8575 \cdot 10^{-18} s^{-1}, \\ \Omega_{\Lambda\tau} = 0.9452, \quad \Omega_{k\tau} = 0, \quad \Omega_{m\tau} = 0.05457, \quad \Omega_{r\tau} = 4.6397 \cdot 10^{-6}, \end{gathered} \tag{11}$$

while the time for this event is computed as

$$\tau = \frac{1}{H_\tau} \int_0^1 \frac{s\,ds}{\sqrt{\Omega_{r\tau} + \Omega_{m\tau} s + \Omega_{k\tau} s^2 + \Omega_{\Lambda\tau} s^4}} = 24.944\,\mathrm{Gyr}. \tag{12}$$

Transition from radiation to matter dominated era. This is more sophisticated example. Radiation dominated epoch covers the period when the expansion of the universe was dominated by radiation. It is usually taken that it started after inflation and lasted until the equalization of matter and radiation. This second event is characterized by

$$\Omega_{m\tau} = \Omega_{r\tau}, \quad \tau \text{ is the equalization time moment.} \tag{13}$$

So we get the ninth equation that supplements the system (7).

In the radiation era, neutrinos were relativistic particles (see Supernova Cosmology Project site), in fact until the recombination which happened far beyond τ. Hence we have to relocate the summand $\Omega_{\nu\tau}$ from $\Omega_{m\tau}$ to $\Omega_{r\tau}$. Therefore, to compute matter density at time τ, instead of

$$\Omega_{m0} = \Omega_{c0} + \Omega_{b0} + \Omega_{\nu 0},$$

we take $\Omega'_{m0} = \Omega_{c0} + \Omega_{b0}$ and $\Omega'_{m\tau} = \Omega'_{m0} x^3 y$ in (7), where $\Omega'_{m\tau} = \Omega_{c\tau} + \Omega_{b\tau}$. Also, as neutrinos at time τ add to the radiation we take $\Omega'_{r\tau} = \Omega_{\gamma\tau} + \Omega_{\nu\tau}$, where $\Omega_{\gamma\tau}$ is the photon density. Hence, instead of the equation (13) we take $\Omega'_{m\tau} = \Omega'_{r\tau}$. Density of neutrinos $\Omega_{\nu\tau}$ at time τ is computed by, see e.g. Gerbino and Lattanzi, Lesgourgues and Pastor:

$$\Omega_{\nu\tau} = \lambda \Omega_{\gamma\tau}, \quad \text{where} \quad \lambda = N_{\text{eff}} \cdot \frac{7}{8} \cdot \left(\frac{4}{11}\right)^{\frac{4}{3}}. \tag{14}$$

Here, $N_{\text{eff}} = 3.046$ is a slightly greater than $N_\nu = 3$, the number of neutrino families. Hence, $\Omega'_{r\tau} = (1+\lambda)\Omega_{\gamma\tau} = (1+\lambda)\Omega_{\gamma 0} x^4 y$. Obviously, we took $\Omega_{r0} = \Omega_{\gamma 0}$, as present neutrinos are non-relativistic, hence they do not add to the radiation. As $\Omega'_{m\tau} = \Omega'_{r\tau}$, we get $\Omega'_{m0} x^3 y = (1+\lambda)\Omega_{\gamma 0} x^4 y$, so

$$x = \frac{1}{1+\lambda} \cdot \frac{\Omega'_{m0}}{\Omega_{\gamma 0}} = \frac{\Omega_{c0} + \Omega_{b0}}{(1+\lambda)\Omega_{\gamma 0}} = \frac{\Omega_{m0} - \Omega_{\nu 0}}{(1+\lambda)\Omega_{\gamma 0}}. \tag{15}$$

As the values of upper and lower bounds and the mean values of the constants Ω_{c0}, Ω_{b0}, $\Omega_{\gamma 0}$ and $\Omega_{\nu 0}$ are known, see the enclosed table, we can solve the system (7) for both values H'_0 and H''_0 (which reflect the Hubble tension). Using the integral (1), or $\mathcal{J}(a)$, we can compute the corresponding times τ. We computed the value $\tau \approx 50\,000$ Yrs which is approximately the same as in Ryden.

Initial data for our computation are displayed in the following table:

Table 1: Values are generated using data from *Particle Data Group*.

Present values of cosmological parameters							
	H_0'	H_0''	$\Omega_{\Lambda 0}$	Ω_{k0}	Ω_{m0}	Ω_{r0}	$\Omega_{\nu 0}$
min	66.9	72.0	0.678	-0.0012	0.3079	$5.23 \cdot 10^{-5}$	0.0012
mean	67.4	73.0	0.685	0.0007	0.3164	$5.38 \cdot 10^{-5}$	0.0021
max	67.9	74.0	0.692	0.0026	0.3249	$5.53 \cdot 10^{-5}$	0.0030

4. CONCLUSION

Algebraic dependencies among the main cosmological parameters are presented. These dependencies can be useful in a uniform approach in computing certain events in the evolution of the universe. We illustrate this usefulness by applying them in computing the age of the universe, the time when the expansion of the universe should be double, and transition moment from radiation to matter dominated era.

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