Publ. Astron. Obs. Belgrade No. 100 (2021), 85 - 96 *Invited Lecture – special session "The gravitational-wave Universe" supported by COST Action CA16104*

NONCOMMUTATIVE $SO(2,3)$ MODEL

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Abstract. This is a review of some of our recent work concerning Noncommutative (NC) Field Theory based on $SO(2,3)_\star$ gauge invariance. An important feature of this theory is that gravitational field, given in terms of a vierbein (frame field), becomes manifest only after a suitable gauge fixing, and it is formally unified with other gauge fields of the theory. Starting with a model of pure NC gravity, we extend it by introducing matter and (non)Abelian gauge fields. Using the enveloping algebra approach and the Seiberg-Witten map, we construct the corresponding NC deformed actions and expand them perturbatively in powers of the canonical parameter of noncommutativity. The first non-vanishing NC correctionturns out to be linear in the NC parameter and it encodes a particular coupling of matter and gauge fields to gravity due to spacetime noncommutativity. This feature is augmented by the fact that some of these corrections pertain even in flat spacetime where they induce potentially observable NC effects. We discuss the obtained NC deformation of electron's dispersion relation in the presence of constant background magnetic field – NC Landau levels.

1. INTRODUCTION

Noncommutative (NC) Field Theory, i.e. a theory of (relativistic) fields on a noncommutative spacetime, is a candidate for an effective theory of the underlying (and yet unknown) fundamental theory of quantum gravity. The construction of a NC field theory relies on the method of deformation quantization via NC $\star$-product (a method also used in phase space quantum mechanics), developed mainly by Flato, Sternheimer and Kontsevich Bayen et al. (1978), Sternheimer (1998), Kontsevich (2013). In general, one speaks of a deformation of an object/structures whenever there is a family of similar objects/structures and a deformation parameter that measures their distortion from the original, undeformed one. In physics, this parameter appears as some fundamental constant of nature that measures a deviation from the classical (undeformed) theory. When it is zero, the classical theory is restored. To deform a continuous structure of spacetime, an abstract algebra of NC coordinates is introduced. These NC coordinates, denoted by $\hat{x}^\mu$, satisfy some non-trivial commutation

[1] Talk given by Voja Radovanović.

relations, and so, it is no longer the case that $\hat{x}^\mu \hat{x}^\nu = \hat{x}^\nu \hat{x}^\mu$. Abandoning this basic property results in various new physical effects that are not present in a field theory developed on ordinary spacetime. The simplest case of noncommutativity is the so called canonical noncommutativity, defined by

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}, \tag{1.1}$$

where $\theta^{\mu\nu}$ are components of a constant, antisymmetric matrix.

Instead of deforming the algebra of coordinates, one can take an alternative, but equivalent approach in which noncommutativity appears in the form of NC $\star$-products of functions (fields) of ordinary commutative coordinates. Specifically, to establish canonical noncommutativity, we use the Moyal-Weyl $\star$-product

$$(\hat{f} \star \hat{g})(x) = e^{\frac{i}{2}\,\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}} f(x)g(y)|_{y\to x}. \tag{1.2}$$

The first term in the expansion is the ordinary point-wise multiplication of functions. The quantities $\theta^{\mu\nu}$ are assumed to be small deformation parameters that have dimensions of $(length)^2$. They are fundamental constants, like the Planck length or the speed of light.

The subject of NC gravity has received a lot of attention and various approaches to this problem have been developed. In Chamseeddine (2001), Chamseeddine (2004), Cardella and Zanon (2003) a deformation of pure Einstein gravity via Seiberg-Witten map is proposed. Twist approach to NC gravity is explored in Aschieri et al. (2005, 2006), Ohl and Schenckel (2009), Aschieri and Castellani (2010). Some other variants are given in Yang (2009), Steinacker (2010), Burić and Madore (2008, 2014), Tomassini and Viaggiu (2014), Faizal (2013), Kobakhidze et al. (2016), Klammer and Steinacker (2009), Harikumar and Rivelles (2006), Dobrski (2017, 2011), Burić et al. (2006b, 2008). The connection to Supergravity (SUGRA) is established in Aschieri and Castellani (2009b), Castellani (2013). Finally, in Dimitrijević Ćirić et al. (2017a), Dimitrijević Ćirić et al. (2017b), Dimitrijević et al. (2012), Dimitrijević and Radovanović (2014) an approach based on canonically deformed Anti de Sitter (AdS) symmetry group, i.e. $SO(2,3)_\star$ group, is developed. In this approach NC gravity is treated as a (deformed) gauge theory, and gravity becomes manifest only after a suitable symmetry breaking (gauge fixing). The action is constructed without previously introducing the metric tensor (it is topological) and the second order NC correction to the Einstein-Hilbert action can be found explicitly. Special attention is devoted to the meaning of coordinates in the context of spacetime noncommutativity. Namely, the results suggest that coordinates in which one postulates canonical noncommutativity are the Fermi inertial coordinates, i.e. coordinates of an inertial observer along a geodesic. Commutator between arbitrary coordinates can in principle be derived from the canonical noncommutativity as demonstrated in Dimitrijević Ćirić et al. (2017a).

The success of the pure gravity model led us to consider matter and gauge fields in the $SO(2,3)_\star$ framework. Dirac spinor field and $U(1)$ gauge field coupled to gravity are introduced in Gočanin and Radovanović (2018) and Dimitrijević-Ćirić et al. (2018), respectively, and physical consequences such as NC deformation of Landau levels are analyzed. From a different perspective, the problem was also treated by

Aschieri and Castellani Aschieri and Castellani (2009a, 2012, 2013), Aschieri (2014). Here we will present the most important results concerning the $SO(2,3)_\star$ framework.

2. COMMUTATIVE MODEL AND ITS NC DEFORMATION

The first step in our analysis is to establish a well-defined commutative (undeformed) model that will subsequently be deformed by substituting ordinary point-wise field multiplication with NC Moyal-Weyl $\star$-product. We propose a commutative action built out of commutative fields on 4-dimensional Minkowski space, endowed with local $SO(2,3)$ symmetry, which can be extended to $SO(2,3) \otimes SU(N)$ if we want to include Yang-Mills fields. For electromagnetic field we use Abelian $U(1)$ group. Then, we demonstrate that, by choosing a suitable gauge (symmetry breaking), this "symmetric-phase" topological action exactly reduces to the action for classical electrodynamics in curved spacetime with the usual, undeformed $SO(1,3) \otimes U(1)$ gauge symmetry.

2. 1. ADS ALGEBRA

Generators of $SO(2,3)$ group are denoted by M_{AB} (with group indices $A, B = 0, 1, 2, 3, 5$) and they satisfy the AdS algebra,

$$[M_{AB}, M_{CD}] = i(\eta_{AD}M_{BC} + \eta_{BC}M_{AD} - \eta_{AC}M_{BD} - \eta_{BD}M_{AC}), \tag{2.3}$$

where η_{AB} is $5D$ flat metric with signature $(+,-,-,-,+)$. A realization of this algebra can be obtained from $5D$ gamma matrices Γ_A that satisfy Clifford algebra $\{\Gamma_A, \Gamma_B\} = 2\eta_{AB}$. The generators are given by $M_{AB} = \frac{i}{4}[\Gamma_A, \Gamma_B]$. One choice of $5D$ gamma matrices is $\Gamma_A = (i\gamma_a\gamma_5, \gamma_5)$, where γ_a are the usual $4D$ gamma matrices. The local Lorentz indices, $a, b, \ldots$, take values $0, 1, 2, 3$. In this particular representation, the $SO(2,3)$ generators are given by $M_{ab} = \frac{i}{4}[\gamma_a, \gamma_b] = \frac{1}{2}\sigma_{ab}$ and $M_{5a} = \frac{1}{2}\gamma_a$. The total gauge potential (master potential) Ω_μ of the $SO(2,3)\otimes U(1)$ gauge group consists of two independent parts, $\Omega_\mu = \omega_\mu + A_\mu$. The first part is the $SO(2,3)$ gauge potential that can be naturally decomposed into spin-connection ${\omega_\mu}^{ab}$ and vierbein e^a_μ,

$$\omega_\mu = \frac{1}{2}{\omega_\mu}^{AB}M_{AB} = \frac{1}{4}{\omega_\mu}^{ab}\sigma_{ab} - \frac{1}{2l}e^a_\mu\gamma_a, \tag{2.4}$$

where l is a constant length scale (AdS radius). Note that in this framework the vierbein field e^a_μ is treated as an additional gauge field, standing on equal footing with the spin-connection (which is a gauge field for the Lorentz group $SO(1,3)$). It is related to the metric tensor by $\eta_{ab}e^a_\mu e^b_\nu = g_{\mu\nu}$ and $e = \det(e^a_\mu) = \sqrt{-g}$. The second part, A_μ, is the electromagnetic field potential. The field strength associated with the full gauge potential Ω_μ is

$$\mathbb{F}_{\mu\nu} = \partial_\mu\Omega_\nu - \partial_\nu\Omega_\mu - i[\Omega_\mu, \Omega_\nu], \tag{2.5}$$

and it can be decomposed as $\mathbb{F}_{\mu\nu} = F_{\mu\nu} + \mathcal{F}_{\mu\nu}$, where the $SO(2,3)$ field strength $F_{\mu\nu}$ is given by

$$F_{\mu\nu} = \partial_\mu\omega_\nu - \partial_\nu\omega_\mu - i[\omega_\mu, \omega_\nu] = \Big({R_{\mu\nu}}^{ab} - \frac{1}{l^2}(e^a_\mu e^b_\nu - e^b_\mu e^a_\nu)\Big)\frac{\sigma_{ab}}{4} - \frac{1}{l}{T_{\mu\nu}}^a\frac{\gamma_a}{2}, \tag{2.6}$$

where $R_{\mu\nu}{}^{ab}$ is the curvature tensor and $T_{\mu\nu}{}^{a}$ torsion. Finally, $\mathcal{F}_{\mu\nu}$ is the usual $U(1)$ field strength

$$\mathcal{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2.7}$$

A necessary step for obtaining electrodynamics in curved spacetime from $SO(2,3) \otimes U(1)$ model is gauge fixing, i.e. symmetry breaking from $SO(2,3)$ to $SO(1,3)$. For that reason one usually introduces an auxiliary field $\phi = \phi^A \Gamma_A$ as in Stelle and West (1980), MacDowell and Mansouri (1977), Towsend (1977), Wilczek (1998). We break the symmetry by fixing the value of the auxiliary field, in particular, by setting $\phi^a = 0$ and $\phi^5 = l$. This field is a spacetime scalar and an internal-space vector and it satisfies the constraint $\phi^A \phi_A = l^2$. It transforms in the adjoint representation of $SO(2,3)$ and its covariant derivative is given by

$$D_\mu \phi = \partial_\mu \phi - i[\Omega_\mu, \phi] = \partial_\mu \phi - i[\omega_\mu, \phi]. \tag{2.8}$$

We see that $U(1)$ part of the master potential doesn't contribute at the classical level. This simplification is a peculiarity of the Abelian $U(1)$ group and it does not hold in a more general case of a non-Abelian Yang-Mills theory. After the gauge fixing, the components of $D_\mu \phi$ become $(D_\mu \phi)^a = e^a_\mu$ and $(D_\mu \phi)^5 = 0$. This is how we get gravity from the auxiliary field ϕ.

2. 2. PURE GRAVITY

In the papers of Stelle, West and Wilczek Stelle and West (1980), Wilczek (1998) a commutative action for *pure gravity* with $SO(2,3)$ gauge symmetry was constructed. Also, in the papers of Chamseddine and Mukhanov Chamseddine and Mukhanov (2013, 2010), GR is formulated by gauging $SO(1,4)$ or, more suitable for SUGRA, $SO(2,3)$ group. Proceeding within this general framework, we show that it can also accommodate fermionic matter fields, specifically, the Dirac spinor field, and electromagnetic $U(1)$ gauge field. We are going to do that by providing a model of commutative action for the Dirac spinors and $U(1)$ gauge field, invariant under ordinary (undeformed) $SO(2,3) \otimes U(1)$ gauge transformations, which exactly reproduces classical electrodynamics in curved spacetime after the symmetry breaking.

In Dimitrijević Ćirić et al. (2017b) the $SO(2,3)$ model of pure gravity action and its NC deformation are analyzed. The commutative action consists of three parts,

$$\begin{aligned}
S_1 &= \frac{ilc_1}{64\pi G_N} \mathrm{Tr} \int d^4x \, \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \phi, && (2.9)\\
S_2 &= \frac{c_2}{128\pi G_N l} \mathrm{Tr} \int d^4x \, \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} D_\rho \phi D_\sigma \phi \phi + h.c., && (2.10)\\
S_3 &= -\frac{ic_3}{128\pi G_N l} \mathrm{Tr} \int d^4x \, \epsilon^{\mu\nu\rho\sigma} D_\mu \phi D_\nu \phi D_\rho \phi D_\sigma \phi \phi. && (2.11)
\end{aligned}$$

Gauge fixing yields

$$S = \frac{-1}{16\pi G_N} \int d^4x \left(\frac{c_1 l^2}{16} \epsilon^{\mu\nu\rho\sigma} \epsilon_{abcd} R_{\mu\nu}{}^{ab} R_{\rho\sigma}{}^{cd} + \sqrt{-g} \left((c_1 + c_2) R - \frac{6}{l^2} (c_1 + 2c_2 + 2c_3) \right) \right). \tag{2.12}$$

For the sake of generality, three a priori undetermined dimensionless constants are introduced. They can be fixed by some consistency condition. The first part of the action is a topological Gauss-Bonnet term which has no effect on the equations of motion, and so, we can set $c_1 = 0$. The Einstein-Hilbert term requires $c_1 + c_2 = 1$, while the absence of the cosmological constant is ensured by having $c_1 + 2c_2 + 2c_3 = 0$.

After NC deformation and perturbative expansion in powers of $\theta^{\mu\nu}$, it was confirmed that the first order NC correction to the commutative action vanishes. This was an already known result, e.g. Aschieri et al. (2013). The first non-vanishing correction is quadratic in the NC parameter. In the low energy limit, equations of motion for the vierbein and the spin-connection are given by

$$\delta e^a_\mu : R_{\alpha\gamma}{}^{cd} e^\gamma_d e^\alpha_a e^\mu_c - \frac{1}{2} e^\mu_a R + \frac{3}{l^2}(1 + c_2 + 2c_3) e^\mu_a = \tau_a{}^\mu = -\frac{8\pi G_N}{e} \frac{\delta S^{(2)}_{NC}}{\delta e^a_\mu}, \quad (2.13)$$

$$\delta\omega_\mu{}^{ab} : T_{ac}{}^c e^\mu_b - T_{bc}{}^c e^\mu_a - T_{ab}{}^\mu = S_{ab}{}^\mu = -\frac{16\pi G_N}{e} \frac{\delta S^{(2)}_{NC}}{\delta\omega_\mu{}^{ab}}. \quad (2.14)$$

The effective energy-momentum tensor $\tau_a{}^\mu$ and the effective spin-tensor $S_{ab}{}^\mu$ both depend on $\theta^{\mu\nu}$, and we can conclude that noncommutativity acts as a source of curvature and torsion. From (2.13) it follows that the scalar curvature of NC Minkowski space is $R = \frac{11}{l^6}\theta^2$. Thus, in $SO(2,3)_\star$ model, there exists an NC deformation of the Minkowski space and the NC correction to the flat metric is given by

$$\begin{aligned} g_{00} &= 1 - R_{0m0n} x^m x^n, \\ g_{0i} &= -\frac{2}{3} R_{0min} x^m x^n, \;\; g_{ij} = -\delta_{ij} - \frac{1}{3} R_{imjn} x^m x^n. \end{aligned} \quad (2.15)$$

Its form suggests that the coordinates x^μ we started with are actually the Fermi normal coordinates. These are inertial coordinates of an observer moving along a geodesic and they can be introduced in a small neighborhood along the geodesic (inside a small cylinder surrounding the geodesic) Manasse and Misner (1963), Chicone and Mashoon (2006), Klein and Randles (2011). The apparent breaking of the diffeomorphism invariance due to canonical noncommutivity can be understood as a consequence of working in a preferred reference system given by the Fermi normal coordinates. A local observer moving along a geodesic measures $\theta^{\mu\nu}$ to be constant. In any other reference frame this will not be the case.

2. 3. COMMUTATIVE ACTIONS FOR MATTER FIELDS

Now we turn to the construction of $SO(2,3) \otimes U(1)$ invariant theory for matter and $U(1)$ gauge field coupled to gravity. In Dimitrijević-Ćirić et al. (2018) we proposed an action for the $U(1)$ gauge field,

$$S_A = c\, Tr \int d^4x\, \varepsilon^{\mu\nu\rho\sigma} \Big\{ f \mathbb{F}_{\mu\nu} D_\rho\phi D_\sigma\phi\phi + \frac{i}{3!} f f D_\mu\phi D_\nu\phi D_\rho\phi D_\sigma\phi\phi \Big\} + h.c. \quad (2.16)$$

It includes an additional auxiliary field $f = \frac{1}{2} f^{AB} M_{AB}$. Like ϕ, this field transforms in the adjoint representation of $SO(2,3)$ and it is invariant under $U(1)$ (i.e. not charged). Its role is to produce the canonical kinetic term for $U(1)$ gauge field in curved spacetime in the absence of the Hodge dual operation (this operation cannot be defined without prior knowledge of the metric tensor, and we don't have one at our disposal). Note also that c is some yet undetermined constant of mass dimension 1.

After gauge fixing, the purely gravitational part of the action (2.16) vanishes and we are left with

$$S_A = -8cl\int d^4x\ e\ f^{ab}e^\mu_a e^\nu_b \mathcal{F}_{\mu\nu} - 4cl\int d^4x\ e\left(f^{ab}f_{ab} + 2f^{a5}f_a{}^5\right). \tag{2.17}$$

Equations of motion for the components of the auxiliary field f are

$$f_{a5} = 0, \quad f_{ab} = -e^\mu_a e^\nu_b \mathcal{F}_{\mu\nu}. \tag{2.18}$$

We use these equations to eliminate the auxiliary field in the action (2.17) and this leaves us with the action for pure $U(1)$ gauge field in curved spacetime,

$$S_A = 4cl\int d^4x\ e\ \mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}. \tag{2.19}$$

To obtain the canonical kinetic term we set $c = -\frac{1}{16l}$.

The Dirac spinor field ψ transforms in the fundamental representation of $SO(2,3)\otimes U(1)$ gauge group. Its covariant derivative is given by (we assume $q = -1$)

$$D_\mu\psi = \partial_\mu\psi - i\Omega_\mu\psi = \nabla_\mu\psi + \frac{i}{2l}e^a_\mu\gamma_a\psi - iA_\mu\psi = \widetilde{\nabla}_\mu\psi + \frac{i}{2l}e^a_\mu\gamma_a\psi, \tag{2.20}$$

where we introduced $\widetilde{\nabla}_\mu = \nabla_\mu - iA_\mu$ as a covariant derivative for $SO(1,3)\otimes U(1)$ gauge group.

In Gočanin and Radovanović (2018) we proposed the following fermionic action

$$S_{\psi,kin} = \frac{i}{12}\int d^4x\ \varepsilon^{\mu\nu\rho\sigma}\Big[\bar\psi D_\mu\phi D_\nu\phi D_\rho\phi D_\sigma\psi - D_\sigma\bar\psi D_\mu\phi D_\nu\phi D_\rho\phi\psi\Big]. \tag{2.21}$$

After the symmetry braking it becomes

$$S_{\psi,kin} = \frac{i}{2}\int d^4x\ e\ \Big[\bar\psi\gamma^\sigma\widetilde{\nabla}_\sigma\psi - \widetilde{\nabla}_\sigma\bar\psi\gamma^\sigma\psi\Big] - \frac{2}{l}\int d^4x\ e\ \bar\psi\psi, \tag{2.22}$$

which is exactly the Dirac action in curved spacetime for spinors with cosmological mass term $2/l$, interacting via $U(1)$ gauge field. Note that $\widetilde{\nabla}_\sigma\psi = \nabla_\sigma\psi - iA_\sigma\psi$, includes the $U(1)$ gauge field.

To obtain fermions with arbitrary mass, we have to include the following additional "mass terms" (terms of the type $\bar\psi\ldots\psi$)

$$\begin{aligned}
S^{(1)}_{\psi,m} &= \frac{ic_1}{2}\Big(\frac{m}{l} - \frac{2}{l^2}\Big)\int d^4x\ \varepsilon^{\mu\nu\rho\sigma}\Big[\bar\psi D_\mu\phi D_\nu\phi D_\rho\phi D_\sigma\phi\phi\psi + \bar\psi\phi D_\mu\phi D_\nu\phi D_\rho\phi D_\sigma\phi\psi\Big],\\
S^{(2)}_{\psi,m} &= \frac{ic_2}{2}\Big(\frac{m}{l} - \frac{2}{l^2}\Big)\int d^4x\ \varepsilon^{\mu\nu\rho\sigma}\Big[\bar\psi D_\mu\phi D_\nu\phi D_\rho\phi\phi D_\sigma\phi\psi + \bar\psi D_\mu\phi\phi D_\nu\phi D_\rho\phi D_\sigma\phi\psi\Big],\\
S^{(3)}_{\psi,m} &= ic_3\Big(\frac{m}{l} - \frac{2}{l^2}\Big)\int d^4x\ \varepsilon^{\mu\nu\rho\sigma}\ \bar\psi D_\mu\phi D_\nu\phi\phi D_\rho\phi D_\sigma\phi\psi.
\end{aligned} \tag{2.23}$$

If we demand that a priori undetermined dimensionless coefficients c_1, c_2, and c_3 satisfy the constraint $c_2 - c_1 - c_3 = -\frac{1}{24}$, after the symmetry breaking, the sum of the three terms in (2.23) becomes

$$S_{\psi,m} = -\left(m - \frac{2}{l}\right)\int d^4x\, e\, \bar{\psi}\psi, \tag{2.24}$$

and the total action, $S_\psi = S_{\psi,kin} + S_{\psi,m}$, is exactly the Dirac action for spinors of mass m in curved spacetime,

$$S_\psi = \frac{i}{2}\int d^4x\, e\, \left[\bar{\psi}\gamma^\sigma\widetilde{\nabla}_\sigma\psi - \widetilde{\nabla}_\sigma\bar{\psi}\gamma^\sigma\psi\right] - m\int d^4x\, e\, \bar{\psi}\psi. \tag{2.25}$$

Thus, by starting with a theory with $SO(2,3)\otimes U(1)$ gauge symmetry, by a suitable gauge fixing, we have obtained the standard action for electrodynamics in curved spacetime.

2. 4. SEIBERG-WITTEN MAP

Now that we have established our commutative $SO(2,3)\otimes U(1)$ model for matter and $U(1)$ gauge field coupled to gravity, we want to deform it via NC Moyal-Weyl $\star$-product. Due to noncommutativity of the $\star$-product NC fields do not belong to the Lie algebra of the gauge group, since the deformed Lie algebra commutation relations do not close in the Lie algebra itself. These fields actually belong to the enveloping algebra. The closure condition for the algebra of gauge transformation becomes a set of differential equations, which are solved by iteration, order-by-order in the NC parameter $\theta^{\alpha\beta}$. Seiberg-Witten (SW) map Jurčo et al. (2001), Seiberg and Witten (1999) provides a solution to these equations. It also ensures that no additional degrees of freedom (no new fields) are included through NC deformation. NC quantities can be represented as perturbation series in powers of the parameter of noncommutativity, with expansion coefficients built out of commutative fields. For example, NC spinor and adjoint field are represented as

$$\widehat{\psi} = \psi - \frac{1}{4}\theta^{\alpha\beta}\omega_\alpha(\partial_\beta + D_\beta)\psi + \mathcal{O}(\theta^2), \tag{2.26}$$

$$\widehat{\phi} = \phi - \frac{1}{4}\theta^{\alpha\beta}\{\omega_\alpha, (\partial_\beta + D_\beta)\phi\} + \mathcal{O}(\theta^2), \tag{2.27}$$

where ω_α is the ordinary gauge potential. We see that at the zeroth order these NC fields reduce to their undeformed counterparts. The obtained NC action possesses deformed $SO(2,3)_\star \otimes U(1)_\star$ symmetry. Assuming that deformation parameter is small, we expend the NC action in powers of $\theta^{\mu\nu}$ using the general formula

$$\begin{aligned}\left(\widehat{A}\star\widehat{B}\right)^{(1)} &= -\frac{1}{4}\theta^{\alpha\beta}\{\omega_\alpha, (\partial_\beta + D_\beta)AB\} + \frac{i}{2}\theta^{\alpha\beta}D_\alpha A D_\beta B + cov(\widehat{A}^{(1)})B \\ &\quad + A cov(\widehat{B}^{(1)}),\end{aligned} \tag{2.28}$$

where $cov(\widehat{A}^{(1)})$ is the covariant part of $A's$ first order NC correction, and $cov(\widehat{B}^{(1)})$, the covariant part of $B's$ first order NC correction. SW ensures the $SO(2,3)\otimes U(1)$

invariance of the expansion, order by order in $\theta^{\alpha\beta}$. It turns out that the leading term in the expansion does not vanish after the symmetry breaking, and we thus obtain linear NC correction to classical electrodynamics in curved spacetime. The calculation is long and tedious and we will not present the details here. Schematically, the spinorial piece is given by

$$\widehat{S}^{(1)}_{\psi} = \theta^{\alpha\beta} \int d^4x \, e \, \bar{\psi}\Big(\mathcal{A}_{\alpha\beta}{}^{\rho\sigma} \widetilde{\nabla}_\rho \widetilde{\nabla}_\sigma + \mathcal{B}_{\alpha\beta}{}^{\sigma} \widetilde{\nabla}_\sigma + \mathcal{C}_\alpha \widetilde{\nabla}_\beta + \mathcal{D}_{\alpha\beta} \Big)\psi. \tag{2.29}$$

Objects $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are complicated functions of geometric and $U(1)$ quantities, e.g. we have interactions of the following type, $\bar{\psi}\sigma_\alpha{}^\sigma \widetilde{\nabla}_\beta \widetilde{\nabla}_\sigma \psi$, $\bar{\psi} R_{\alpha\beta}{}^{\rho\sigma} \gamma_\rho \widetilde{\nabla}_\sigma \psi$, $\bar{\psi} T_{\alpha\beta}{}^{\sigma} \widetilde{\nabla}_\sigma \psi$, $\bar{\psi}\mathcal{F}_{\alpha\mu}\gamma^\mu \widetilde{\nabla}_\beta \psi$, $\bar{\psi}\mathcal{F}_{\alpha\beta}\psi$, $\bar{\psi}\sigma_{\alpha\beta}\psi$ etc. More importantly, we want to emphasize the fact that this θ-linear NC correction pertains in the limit of flat spacetime. This enables us to derive some tangible phenomenological consequences of our model that could potentially be tested experimentally in a not to far future.

3. FLAT SPACETIME NC ELECTRODYNAMICS

The action for NC electrodynamics in flat spacetime, up to first order in $\theta^{\alpha\beta}$, is given by

$$\begin{aligned} \widehat{S}_{flat} = & \int d^4x \Big[\bar{\psi}(i\not{\mathcal{D}} - m)\psi - \frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} \Big] - \theta^{\alpha\beta} \int d^4x \, \Big[\frac{1}{8}\mathcal{F}_{\alpha\beta}\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu} + \frac{1}{2}\mathcal{F}_{\alpha\mu}\mathcal{F}_{\beta\nu}\mathcal{F}^{\mu\nu} \Big] \\ & + \theta^{\alpha\beta} \int d^4x \, \bar{\psi} \Big[-\frac{1}{2l}\sigma_\alpha{}^\sigma \mathcal{D}_\beta \mathcal{D}_\sigma + \frac{7i}{24l^2}\varepsilon_{\alpha\beta}{}^{\rho\sigma}\gamma_\rho\gamma_5\mathcal{D}_\sigma - \Big(\frac{m}{4l^2} + \frac{1}{6l^3}\Big)\sigma_{\alpha\beta} \\ & + \frac{3i}{4}\mathcal{F}_{\alpha\beta}\not{\mathcal{D}} - \frac{i}{2}\mathcal{F}_{\alpha\mu}\gamma^\mu\mathcal{D}_\beta - \Big(\frac{3m}{4} - \frac{1}{4l}\Big)\mathcal{F}_{\alpha\beta} \Big]\psi, \end{aligned} \tag{3.30}$$

where we introduced flat spacetime covariant derivative $\mathcal{D}_\mu = \partial_\mu - iA_\mu$. We notice immediately that this action is different than the actions for NC electrodynamics already present in the literature Burić and Radovanović (2002), Wulkenhaar (2002), Burić et al. (2006a). The new interaction terms between spinors and the $U(1)$ field, specific to the $SO(2,3)_\star$ model, appear as residuals from the gravitational interaction, and they lead to some non-trivial consequences such as the modification of the dispersion relation for electrons.

By varying the NC action (3.30) with respect to $\bar{\psi}$ we obtain deformed Dirac equation for electron coupled to some background electromagnetic field A_μ,

$$\big(i\not{\partial} - m + \not{A} + \theta^{\alpha\beta}\mathcal{M}_{\alpha\beta}\big)\psi = 0. \tag{3.31}$$

To simplify the analysis, we assume that only two spatial dimensions are mutually incompatible, e.g. $[x^1, x^2] = i\theta^{12}$. Thus, we have $\theta^{12} = -\theta^{21} =: \theta \neq 0$ and all other components of $\theta^{\alpha\beta}$ equal to zero. We will consider the case of an electron in background magnetic field.

3. 1. ELECTRON IN BACKGROUND MAGNETIC FIELD

Using the NC deformed Dirac equation (3.31), we can see how noncommutativity modifies the energy levels of an electron in a constant background magnetic field $\boldsymbol{B} = B\boldsymbol{e}_z$. Classical (undeformed) energy levels for a relativistic electron are given by

$$E_{n,s}^{(0)} = \sqrt{p_z^2 + m^2 + (2n+s+1)B}. \tag{3.32}$$

We are looking for a linear NC correction $E_{n,s}^{(1)} \sim \theta$ of the energy levels (3.32). Working perturbatively in θ, they can be calculated by the following formula,

$$E_{n,s}^{(1)} = -\frac{\theta^{\alpha\beta} \int dy\; \bar{\psi}_{n,s}^{(0)} \mathcal{M}_{\alpha\beta} \psi_{n,s}^{(0)}}{\int dy\; \bar{\psi}_{n,s}^{(0)} \gamma^0 \psi_{n,s}^{(0)}}. \tag{3.33}$$

In particular, for $\theta^{12} = -\theta^{21} = \theta \neq 0$ we obtain

$$\begin{aligned} E_{n,s}^{(1)} =& -\frac{\theta s}{E_{n,s}^{(0)}}\left[\frac{m^2}{12l^2} - \frac{m}{3l^3}\right] - \frac{\theta B s}{E_{n,s}^{(0)}(E_{n,s}^{(0)}+m)}\left[\frac{m}{12l^2} - \frac{1}{3l^3}\right](2n+s+1) \\ &+ \frac{\theta B^2}{2E_{n,s}^{(0)}}(2n+s+1). \end{aligned} \tag{3.34}$$

Non-relativistic limit of the NC energy levels is obtained by expanding undeformed energy function $E_{n,s}^{(0)}$ assuming p_z^2, $B \ll m^2$,

$$\begin{aligned} E_{n,s}^{(0)} &= \sqrt{p_z^2 + m^2 + (2n+s+1)B} \\ &\approx m\left[1 + \frac{p_z^2 + (2n+s+1)B}{2m^2} - \frac{(p_z^2 + (2n+s+1)B)^2}{8m^4}\right]. \end{aligned} \tag{3.35}$$

Expending (3.34) we obtain the NC correction to the energy levels of a non-relativistic electron,

$$\begin{aligned} E_{n,s}^{(1)} =& \left[\frac{\theta s}{3l^3} - \frac{\theta s m}{12l^2}\right]\left[1 - \frac{p_z^2}{2m^2} + \frac{3p_z^4}{8m^4} + \frac{3p_z^2(2n+s+1)B}{8m^4}\right] \\ &+ \frac{\theta B^2}{2m}(2n+s+1)\left[1 - \frac{p_z^2 + (2n+s+1)B}{2m^2} + \frac{3(p_z + (2n+s+1)B)^2}{8m^4}\right]. \end{aligned} \tag{3.36}$$

For an electron constrained to the NC x,y-plane we take $p_z = 0$ and NC energy levels reduce to

$$\begin{aligned} E_{n,s} =& \left[m - s\theta\left(\frac{m}{12l^2} - \frac{1}{3l^3}\right)\right] + \frac{2n+s+1}{2m}B_{eff} - \frac{(2n+s+1)^2}{8m^3}B_{eff}^2 \\ &+ \mathcal{O}(\theta^2), \end{aligned} \tag{3.37}$$

where we introduced $B_{eff} = (B + \theta B^2)$ as an effective magnetic field. We see that a spin-dependent shift of mass appears. If we compare this expression with the one for undeformed energy levels $E_{n,s}^{(0)}$, we see that the only effect of noncommutaivity is to modify the mass of an electron and the value of the background magnetic field.

This interpretation of constant noncommutativity is in accord with string theory. In the famous paper of Seiberg and Witten Seiberg and Witten (1999) it is argued, in the context of string theory, that coordinate functions of the endpoints of an open string constrained to a D-brane in the presence of a constant Neveu-Schwarz B-field satisfy the constant noncommutativity algebra. This implies is that a relativistic field theory on NC spacetime can be interpreted as a low energy limit of the theory of open strings.

4. FURTHER DEVELOPMENT

This newly established theory of NC Electrodynamics, both in curved and flat spacetime, paves the way for a variety of further investigation. Here we point to some of them:

(1) *Experimental verification*: The $SO(2,3)_\star \otimes U(1)_\star$ model of NC Electrodynamics predicts a potentially observable modifications of some of the basic properties of an electron. It is crucial that these corrections are linear in the noncommutativity parameter. With our growing ability to probe high energies scales, this could enable us to obtain an experimental verification.

(2) *Renormalization*: The so called minimal NC electrodynamics is not a renormalisabile theory because of the fermionic loop contributions Burić and Radovanović (2002), Wulkenhaar (2002), Burić et al. (2006). It would be interesting to analyze the renormalisability of the $SO(2,3)_\star$ model in order to theoretically determine on which scale does noncommutativity operates.

(4) *NC Standard Model*: One could incorporate scalar fields in the $SO(2,3)_\star$ framework in order to construct the full NC extension of the Standard Model of elementary particles.

(6) *NC Quantum Hall Effect*: Quantum Hall Effect is one of the most interesting phenomena in physics. Our results could be extended in a way that would allow us to see how noncommutativity modifies the behavior of electrons in some material medium. One line of investigation is to explore the Quantum Hall Effect taking into account the NC deformation of Landau levels found in our model.

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