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A MODEL OF QUANTUM COSMOLOGY: FUZZY DE SITTER SPACE

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Abstract. Quantization of gravity is probably the most important unsolved problem of theoretical physics today. For many years it has been approached only theoretically, but in the last decades there is a growing amount of astrophysical and cosmological data that give input and guide the further research. Present theories of quantum gravity are formulated in a variety of ways: we here describe a model of fuzzy de Sitter space obtained in the context of noncommutative geometry, and discuss some of its implications to cosmology.

1. INTRODUCTION

In many ways, gravity is unique among physical phenomena. Historically it had a distinctive role in our understanding and description of Nature, as
– observations of motion of planets and Sun established astronomy as first of the natural sciences, while attempts to systematize and explain these observations made gravity the first of fundamental forces to be theoretically described;
– efforts to understand gravity brought, in physics, fundamental ideas like Newton's laws of classical mechanics and Einstein's general relativity (GR);
– Newton's and Einstein's theories marked significant steps in the development of mathematics (differential calculus and analysis, geometry) and its relation to physics;
– finally, a unique feature of gravity is that it has not been quantized, yet. It is indeed this segment where we expect gravity to bring the next fundamental breakthrough.

Some of important GR results that form our current intuition about gravity are:
– spacetime is not an empty stage, a fixed framework (as in Newton's mechanics), but a dynamical object interacting with matter. In particular, the universe evolves;
– gravitational force can be described in purely geometric terms: physical quantities that characterize spacetime are invariants like geodesic lines or curvature invariants;
– dynamics of spacetime is described through a classical field that carries energy and momentum: Einstein equations are obtained from the principle of minimal action;
– Einstein equations are covariant under the changes of coordinates: diffeomorphism invariance can be understood as a guiding symmetry principle to formulate GR;
– solutions to Einstein equations generically have singularities, like the black hole or the big bang singularity. This result (due to Penrose and Hawking, around 1965) was awarded by the Nobel prize in 2020. To obtain it, one analyzes the real physical situations: the energy conditions on matter, boundary conditions, etc.

However, we know that matter is quantized, either as point particles (in the non-relativistic regime) or as quantum fields. This raises two important questions: how to couple classical gravity to quantum matter; further, how to quantize gravity? The standard recipe for quantization in quantum field theory (QFT) is to describe gravity as classical field theory, expand it around the minimal-energy solution (vacuum), and quantize perturbatively. Alternatively, one can generalize the quantum-mechanical approach: write the dispersion relation for gravity (Hamiltonian constraint) and quantize it, to obtain the gravitational analogue of the Schrödinger equation (Wheeler-de Witt equation). But the standard recipes do not work: they give either unphysical results (as gravity is not renormalizable) or formal answers (as there is no effective way to do calculations that include functional equations, beyond the simplest cases). Apart from heuristic reasons, gravity has to be quantized for physical and consistency reasons: if it is a fundamental force, it should be unified with other fundamental interactions and described in the same way, by quantum field theory. Furthermore, classical singularities of GR certainly are unphysical, and quantization is a way to remove them.

There is no *a priori* justification to extrapolate, to assume that spacetime on the Planck scale has the same structure as it has on the atomic, galactic or cosmological scales. Perhaps the (classical) structure of spacetime at small scales is not that of a manifold, but discrete ('quantum spacetime does not have points')? Quantum field theories on flat and curved spaces have the problem of ultraviolet (UV) divergences: the propagator between points x and x' is divergent in the coincidence limit $x \to x'$ (i.e. $p \to \infty$). This problem is solved by renormalization; but were the spacetime structure lattice-like, it would have not existed. Diverse ideas are developed along these lines. Perhaps the most physical way to give structure to the spacetime points is to develop a model that 'delocalizes' their description, as in string theory or in loop quantum gravity. Another, straightforward way, is to describe spacetime coordinates by noncommuting operators, as in quantum mechanics. An important constraint on all theories of quantum gravity is the classical, that is, macroscopic limit to GR. In principle we build from the known: we usually keep one of desirable or intuitive properties of general relativity (e.g. its field-theoretic interpretation, geometric interpretation, symmetry principle) and relax the others.

2. NONCOMMUTATIVE GEOMETRY

The framework which we use is that of noncommutative (NC) geometry. NC geometry is a very active area of research since the 1990s; some prominent names who developed it on the side of mathematics include Fields medalists Connes and Kontsevich. Many theoretical physicists, aspiring to different physical applications, have been involved in this area of research, developing notions of noncommutative space, noncommutative differential geometry and noncommutative field theory. An approach inspired by the geometric description gravity, in the sense that it generalizes the Cartan formalism of differential geometry, is Madore's noncommutative frame formalism (Madore, 1995) used here. We present a model of noncommutative or fuzzy de Sitter space developed in Buric *e*t al. 2015, 2018, 2019.

There are several noncommutative spaces whose properties are thoroughly investigated and well established, including description of classical and quantized fields on them. The best known example is NC space with constant noncommutativity of Cartesian coordinates, or 'Moyal deformation' of the flat space: there are various applications of this model to cosmology, though one should perhaps notice that it is, by definition, anisotropic. Another paradigmatic example is the fuzzy sphere, a two-dimensional spherically symmetric NC space of constant curvature defined by the Lie algebra of rotation group. Further examples are κ-Minkowski space, fuzzy hyperboloid, fuzzy $\mathbb{CP}^n$ spaces, spaces built on different $\star$-products, etc. In order to obtain noncommutative spaces that extend basic solutions of GR (e.g. FLRW cosmologies, black holes), one should preferably keep the spherical symmetry and work in four spacetime dimensions: but this task proves to be far from straightforward (Buric and Madore, 2014). The main reason is that the algebraic structure of spacetime imposes additional constraints which are, beyond commutative geometry, rather nontrivial. We discuss here a generalization of the fuzzy sphere construction: the four-dimensional fuzzy de Sitter space, based on the algebra of the de Sitter group $SO(1,4)$ and its (unitary irreducible) representations. Although the full details are rather technical, let us at least introduce the basic elements of the description.

Noncommutative space has a structure of an algebra $\mathcal{A}$. It is generated by coordinates x^μ which are real i.e. hermitian, but in principle, non-commuting,

$$[x^\mu, x^\nu] = i\bar{k} J^{\mu\nu}(x)\,. \tag{1}$$

They can be ordinary commutative variables but also operators, finite matrices, etc. Dimensional parameter $\bar{k}$ sets the scale of noncommutativity; the formal limit $\bar{k} \to 0$ is the commutative or macroscopic limit mentioned above. Uncertainty relations that follow from (1) in the case when $J^{\mu\nu} \neq 0$ imply that 'there are no points' on a specific NC space, i.e. that all coordinates cannot be measured simultaneously. The structure of a NC space can be understood in terms of the spectra of its coordinates. Obviously, a change of coordinates changes their spectra, but there is an overall diffeomorphism invariance, meaning that one can transform (1) using the standard algebraic rules. Apart from the properties of coordinates, there are other ways to describe a NC space, e.g. its symmetries, its coherent states, and of course its commutative limit.

Differential-geometric structure of $\mathcal{A}$ is given by the momentum algebra, i.e. the algebra of derivatives. In the NC frame formalism, momenta p_α are functions or operators that define the free falling frame (tetrad) e_α,

$$e_\alpha f = [p_\alpha, f]. \tag{2}$$

The commutator satisfies the Leibniz rule, so e_α is a derivation. Dual to derivations e_α are differential 1-forms θ^α; the differential of a function is defined as

$$df = (e_\alpha f)\,\theta^\alpha\,. \tag{3}$$

On curved commutative manifold the moving frame is given by its components e_α^μ, $e_\alpha f = e_\alpha^\mu\,(\partial_\mu f)$, and momenta are combinations of partial derivatives,

$$p_\alpha = e_\alpha^\mu\,\partial_\mu, \qquad e_\alpha^\mu = [p_\alpha, x^\mu]\,. \tag{4}$$

In analogy, in the noncommutative case the tetrad and the metric are defined by

$$e_\alpha^\mu = [p_\alpha, x^\mu]\,, \qquad g^{\mu\nu} = e_\alpha^\mu\, e_\beta^\nu\, \eta^{\alpha\beta}\,, \tag{5}$$

with additional conditions to assure orthonormality of the moving frame. Laplacian of a scalar function is defined as

$$\Delta f = \eta^{\alpha\beta}[p_\alpha, [p_\beta, f]] . \tag{6}$$

It is possible, and rather straightforward, to define quantities like connection, covariant derivative, curvature and torsion, so one can achieve the full differential-geometric description. Further, one can introduce scalar, spinor and gauge fields with their classical equations of motion. Action for the classical fields can be given as well, providing that there is well defined integral/trace.

3. FUZZY DE SITTER SPACE

As a model of quantum cosmology we discuss four-dimensional fuzzy de Sitter space. In the commutative case, de Sitter space is defined as an embedding

$$-v^2 + w^2 + x^2 + y^2 + z^2 = \frac{3}{\Lambda} \tag{7}$$

in the flat 5-dimensional space

$$ds^2 = -dv^2 + dw^2 + dx^2 + dy^2 + dz^2, \tag{8}$$

where Λ is the cosmological constant. De Sitter space is a maximally symmetric space, its symmetry group is the de Sitter group $SO(1,4)$.

In order to introduce the fuzzy version of de Sitter space, it is natural to start with its symmetry algebra. This enables, on the one hand, to control symmetries of the resulting NC space; on the other hand, it allows concrete calculations as there are exact results on this Lie algebra and its representations. The $so(1,4)$ algebra has ten generators $M_{\alpha\beta}$ that satisfy

$$[M_{\alpha\beta}, M_{\gamma\delta}] = -i(\eta_{\alpha\gamma}M_{\beta\delta} - \eta_{\alpha\delta}M_{\beta\gamma} - \eta_{\beta\gamma}M_{\alpha\delta} + \eta_{\beta\delta}M_{\alpha\gamma}) , \tag{9}$$

$\alpha, \beta, ... = 0,1,2,3,4$; our signature is $\eta_{\alpha\beta} = \mathrm{diag}(+----)$. The Casimir operators, quadratic and quartic, are

$$\mathcal{Q} = -\frac{1}{2}\, M_{\alpha\beta}M^{\alpha\beta}, \qquad \mathcal{W} = -W_\alpha W^\alpha , \tag{10}$$

with the 'Pauli-Lubanski' vector $W_\alpha = \frac{1}{8}\,\epsilon_{\alpha\beta\gamma\delta\eta}M^{\beta\gamma}M^{\delta\eta}$. The Casimir relation $\mathcal{W}=$ const is analogous to the embedding (7) which defines commutative de Sitter space. Therefore we introduce noncommutative coordinates as

$$x^\alpha = \ell\, W^\alpha \tag{11}$$

and define fuzzy de Sitter space as a unitary irreducible representation of the $SO(1,4)$.

There are at least two choices of momenta that give geometry (metric, curvature) of the de Sitter space in the commutative limit. We discuss the one defined by four momenta p_0, p_i,

$$ip_0 = \sqrt{\Lambda}\, M_{04} , \qquad ip_i = \sqrt{\Lambda}\,(M_{i4} + M_{0i}). \tag{12}$$

The line element that the noncommutative frame formalism gives is

$$ds^2 = -(\theta^0)^2 + (\theta^i)^2 = -d\tau^2 + e^{2\tau}\, dx^i dx^i \tag{13}$$

and the corresponding scalar curvature is constant. From the expression for the line element we can identify the cosmic time,

$$\frac{\hat{\tau}}{\ell} = \log \frac{x^0 + x^4}{\ell} = \log\,(W^0 + W^4)\,. \tag{14}$$

It is different from the embedding time, $x^0 = \ell W^0$; the spatial coordinates are $x^i = \ell W^i,\ i = 1, 2, 3$.

Unitary irreducible representations of the de Sitter group are infinite-dimensional, labelled two quantum numbers (s, ρ). They fall into three categories (Dixmier, 1961):

∘ principal continuous series: $\rho \geq 0,\ \ s = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$
$\mathcal{Q} = -s(s+1) + \frac{9}{4} + \rho^2, \quad \mathcal{W} = s(s+1)(\frac{1}{4} + \rho^2)$

∘ complementary continuous series: $\nu = i\rho \in \mathbb{R}, \quad |\nu| < \frac{3}{2}, \quad s = 0, 1, 2 \dots$
$\mathcal{Q} = -s(s+1) + \frac{9}{4} - \nu^2, \quad \mathcal{W} = s(s+1)(\frac{1}{4} - \nu^2)$

∘ discrete series: $s = \frac{1}{2}, 1, \frac{3}{2}, 2 \dots, \quad q = \frac{1}{2} + \nu = \frac{1}{2} + i\rho = s, s-1, \dots 0$ or $\frac{1}{2}$
$\mathcal{Q} = -s(s+1) - (q+1)(q-2), \quad \mathcal{W} = -s(s+1)q(q-1)$.

We would like to determine the spectra of the above-defined physical coordinates. The most effective way is to use the Hilbert space representation corresponding to one of representations given above, in which $M_{\alpha\beta}$, W_α are differential operators. Such representations exist: we will discuss the principal continuous series (ρ, $s = \frac{1}{2}$). The representation space is the space of Dirac bispinors $\psi(\vec{p})$ that satisfy the Dirac equation, with scalar product

$$(\psi, \psi) = \int \frac{d^3p}{2p_0}\, \psi^\dagger \gamma^0 \psi\,. \tag{15}$$

Using the Dirac representation of γ-matrices, we find

$$\psi(\vec{p}) = \begin{pmatrix} \varphi(\vec{p}) \\ -\dfrac{\vec{p}\cdot\vec{\sigma}}{p_0 + m}\,\varphi(\vec{p}) \end{pmatrix}, \qquad (\psi, \psi') = \int \frac{d^3p}{p_0}\, \frac{2m}{p_0 + m}\, \varphi^\dagger \varphi'\,, \tag{16}$$

where $\varphi(\vec{p})$ is a spinor and $\vec{\sigma}$ are the Pauli matrices. The group generators are

$$M_{ij} = L_{ij} + S_{ij}, \qquad S_{ij} = \frac{i}{4}\,[\gamma_i, \gamma_j] \tag{17}$$

$$M_{0i} = L_{0i} + S_{0i}, \qquad S_{0i} = \frac{i}{4}\,[\gamma_0, \gamma_i] \tag{18}$$

$$M_{40} = -\frac{\rho}{m}\, p_0 + \frac{1}{2m}\,\{p^i, M_{0i}\}\,, \tag{19}$$

$$M_{4k} = -\frac{\rho}{m}\, p_k - \frac{1}{2m}\,\{p^0, M_{0k}\} - \frac{1}{2m}\,\{p^i, M_{ik}\}\,, \tag{20}$$

with

$$L_{ij} = i\left(p_i\,\frac{\partial}{\partial p^j} - p_j\,\frac{\partial}{\partial p^i}\right), \qquad L_{0i} = ip_0\,\frac{\partial}{\partial p^i}\,, \tag{21}$$

$$L_{40} = -\frac{\rho}{m}\,p_0 + \frac{1}{2m}\,\{p^i, L_{0i}\}\,, \tag{22}$$

$$L_{4k} = -\frac{\rho}{m}\,p_k - \frac{1}{2m}\,\{p^0, L_{0k}\} - \frac{1}{2m}\,\{p^i, L_{ik}\}\,. \tag{23}$$

Using these, we obtain the operators of coordinates in this representation,

$$\frac{x^0}{\ell} = W^0 = -\frac{1}{2m}\begin{pmatrix} (\rho-\frac{i}{2})p_i\sigma^i + i\,p_0^2\,\frac{\partial}{\partial p^i}\sigma^i & \epsilon^{ijk}p_0p_i\,\frac{\partial}{\partial p^j}\,\sigma_k + \frac{3i}{2}p_0 \\ \epsilon^{ijk}p_0p_i\,\frac{\partial}{\partial p^j}\,\sigma_k + \frac{3i}{2}p_0 & (\rho-\frac{i}{2})p_i\sigma^i + ip_0^2\,\frac{\partial}{\partial p^i}\,\sigma^i \end{pmatrix}, \tag{24}$$

$$\frac{x^4}{\ell} = W^4 = -\frac{1}{2}\begin{pmatrix} ip_0\,\frac{\partial}{\partial p^i}\sigma^i & \epsilon^{ijk}p_i\,\frac{\partial}{\partial p^j}\,\sigma_k + \frac{3i}{2} \\ \epsilon^{ijk}p_i\,\frac{\partial}{\partial p^j}\,\sigma_k + \frac{3i}{2} & ip_0\,\frac{\partial}{\partial p^i}\sigma^i \end{pmatrix}. \tag{25}$$

The remaining expression for the x^i is in principle of similar structure but longer.

4. COSMOLOGICAL IMPLICATIONS

We can now formulate and solve the eigenvalue equations for coordinates of the fuzzy de Sitter space and determine their spectra. W^0, W^4 and W^0+W^4 commute with the angular momenta L_i, so we can choose their eigenfunctions in the form

$$\varphi(\vec{p}) = \frac{f(p)}{p}\,\varphi_{jm} + \frac{h(p)}{p}\,\chi_{jm}\,, \tag{26}$$

where the spinor spherical harmonics are given by

$$\varphi_{jm} = \begin{pmatrix} \sqrt{\frac{j+m}{2j}}\,Y_{j-1/2}^{m-1/2} \\ \sqrt{\frac{j-m}{2j}}\,Y_{j-1/2}^{m+1/2} \end{pmatrix}, \quad \chi_{jm} = \begin{pmatrix} \sqrt{\frac{j+1-m}{2(j+1)}}\,Y_{j+1/2}^{m-1/2} \\ -\sqrt{\frac{j+1+m}{2(j+1)}}\,Y_{j+1/2}^{m+1/2} \end{pmatrix}, \tag{27}$$

$Y_l^m(\theta,\varphi)$ are the usual spherical harmonics in momentum space, $p=|\vec{p}|$, etc. The nontrivial part for each eigenvalue problem is the radial equation for $f(p)$ and $h(p)$. Computations are long but relatively straightforward: we just review the results.

Because of the $SO(1,4)$ symmetry, the spectra of spatial coordinates W^i and of W^4 are the same. The spectrum of W^4 is continuous: the real line. Its eigenfunctions $|\lambda jm\rangle$ are normalized as

$$\langle\lambda jm|\lambda' j' m'\rangle = \delta(\lambda-\lambda')\,\delta_{jj'}\,\delta_{mm'}\,. \tag{28}$$

Their radial part contains the associated Legendre functions, $P_{-2i\lambda}^{-j}(p_0)$.

The spectrum of the embedding time W^0 is discrete and its eigenvalues are given by $k(k+1)-k'(k'+1)$, where $k,k'=0,\frac{1}{2},1,\frac{3}{2},\ldots$.

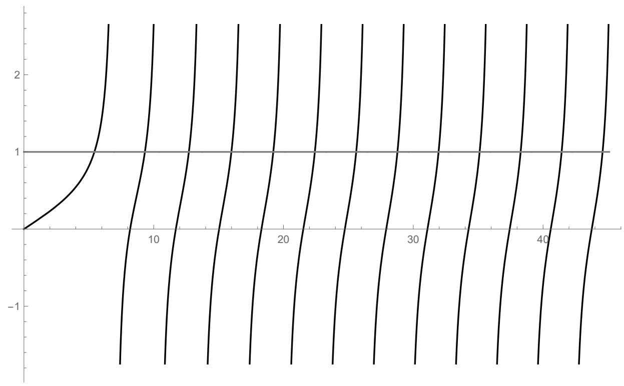
The radial equation for the cosmic time $e^{\hat{\tau}/\ell} = W^0 + W^4$ is the most interesting. It reduces to the Bessel equation in variable $z = \sqrt{\frac{(p_0-1)}{(p_0+1)}}$ and the solutions are given in terms of Bessel functions $J_j(2\lambda z)$. The eigenvalue $\lambda = e^{\tau/\ell}$ is continuous,$\lambda \in (0, \infty)$. This is in contradiction with the fact that, apparently, all eigenfunctions are normalizable. A careful analysis shows that, naively defined, the cosmic time $\hat{\tau}$ is not a self-adjoint operator. To obtain its self-adjoint extension one has to add specific boundary conditions, i.e. to reduce the initial Hilbert space of functions to a subspace of physical states. This reduction makes the spectrum of the cosmic time discrete, and the corresponding eigenfinctions $|\tau jm\rangle$ orthonormal,

$$\langle \tau jm | \tau' j' m' \rangle = \delta_{\tau\tau'}\, \delta_{jj'}\, \delta_{mm'}\, . \tag{29}$$

The boundary condition is of the form

$$\frac{J_{j+1}(2\lambda)}{J_j(2\lambda)} = \text{const}\, , \tag{30}$$

and it determines the allowed discrete eigenvalues, $\tau = \ell \log \lambda$. The solutions for value 1 of the given constant are given graphically below.



Let us summarize and discuss consequences of our results to cosmology.

The main result is that spatial coordinates of fuzzy de Sitter space are continuous, while time is discrete. If we calculate the expectation value of the radius of universe,

$$(x^i)^2 = -\ell^2\, W_i W^i \tag{31}$$

at fixed moment of time τ, we find

$$\mathcal{W} + \lambda^2 \leq \langle \tau jm | -\ell^2\, W_i W^i | \tau jm \rangle \leq \mathcal{W} + 2\lambda^2\, , \tag{32}$$

where $\mathcal{W}$ is the value of the quartic Casimir operator, $\ell\sqrt{\mathcal{W}} = \ell\sqrt{\frac{3}{16} + \frac{3\rho^2}{4}} \geq 0$. This means that the radius of the universe is bounded below by $\ell\sqrt{\mathcal{W}}$: it cannot vanish in physical states, which implies that there is no big bang singularity.

The radius of the universe grows with time exponentially: for late times we have the usual behavior

$$\sqrt{\langle \tau jm| - \ell^2\, W_i W^i |\tau jm\rangle} \sim \lambda = e^{\tau/\ell}. \tag{33}$$

Discreteness of time becomes relevant only in the 'deep quantum region' $\lambda \to 0$, i.e. $\tau \to -\infty$. For values away from the Planck scale time is almost continuous: the difference between its consecutive eigenvalues is macroscopically negligible,

$$\tau_{n+1} - \tau_n \approx \ell\, \log\left(1 + \frac{1}{n}\right). \tag{34}$$

Further interesting properties of our model, not discussed in details here, include symmetry. Namely, the choice of a specific self-adjoint extension breaks the initial symmetry at distances of order ℓ, i.e. near the Planck scale. This 'spontaneous symmetry breaking' can be pursued in all mathematical details; in particular, in the macroscopic limit $\ell \to 0$ the full symmetry is recovered.

There are several possible directions in which this work should be continued. In applications to cosmology, an important problem (well defined within the given formalism) is to include the scalar fied: describe its classical evolution, fluctuations and then eventually, find the implications to inflation. We expect, further, that (small) anisotropies like those observed in the CMB can be described by perturbation theory, and indeed, there are results for perturbations of the flat noncommutative spaces. Of course, for a fuller characterization of the CMB radiation one should develop a description of gauge and fermion fields on the fuzzy de Sitter background. These results would give signatures of noncommutativity, i.e. its potentially measurable effects. A more difficult problem, as we see it, is to find other 'ground states' of noncommutative geometry, NC spaces that describe relevant configurations like an arbitrary FLRW spacetime or black holes. Typically in these cases there is less symmetry, so it is not quite clear what is the algebraic structure one should start with. However, it is an important avenue of further research as, as we have seen, noncommutative geometry can provide mechanisms to solve singularity problems of general relativity while preserving the correct macroscopic limits.

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