Dynamics of Celestial Bodies, 197 - 200 *Contributed paper*

BIFURCATIONS OF PERIODIC ORBITS IN THE GENERAL 3-BODY PLANETARY PROBLEM

G. VOYATZIS and T. KOTOULAS

Department of Physics, Section of Astrophysics, Astronomy and Mechanics, 54 124, Thessaloniki, Greece
E–mail: voyatzis@auth.gr

Abstract. In this paper we present cases of biffurcation of families of periodic orbits within the framework of the general three body problem. We restrict ourselves to the 2/1 and 3/1 resonant cases of two-planet systems and we demonstrate bifurcations which cause critical changes in the structure of families of periodic orbits, and, furthermore, in the topology of phase space, as the planetary mass ratio ρ varies. We consider the whole range $0 < \rho < \infty$ and, therefore, we include the passage from external to internal resonances.

1. INTRODUCTION

It is well known that resonant families of periodic orbits exist for the general three body problem. These families can be obtained either by continuation from a family of circular orbits (Voyatzis and Hadjidemetriou, 2005,2006) or by computing stationary solutions of the averaged resonant Hamiltonian (Beauge et al, 2003; Michtchenko et al, 2006). In a resent work (Voyatzis et al, 2008) we show that all these families can be derived by continuation of the families of the circular and the elliptic restricted problem.

The basic model, which is used here, is the general planar three body problem (TBP) consisting of a star S of mass m_0 and two planets P_1 (inner) and P_2 (outer). The system is given in a rotating frame which reduces the system to three degrees of freedom (Hadjidemetriou, 1975, 2006).

Since $m_1 \ll m_0$ and $m_2 \ll m_0$, the position of periodic orbits depends (in first order) to the planetary mass ratio $\rho = m_2/m_1$. For $\rho = 0$ we get the restricted problem and the case of external resonances (i.e. the massless body moves outside the orbit of the massive planet). For $\rho = \infty$ (but keeping m_2 finite) we get the case of internal resonances of the restricted problem (i.e. the massless body moves inside the orbit of the massive planet). By varying ρ, the existing families of periodic orbits are continued and obey bifurcations at critical values of ρ. Such bifurcations are demonstrated in the present paper for the 2/1 and 3/1 resonances. For each case, details are given in Voyatzis et al. (2008) and Voyatzis (2008), respectively.

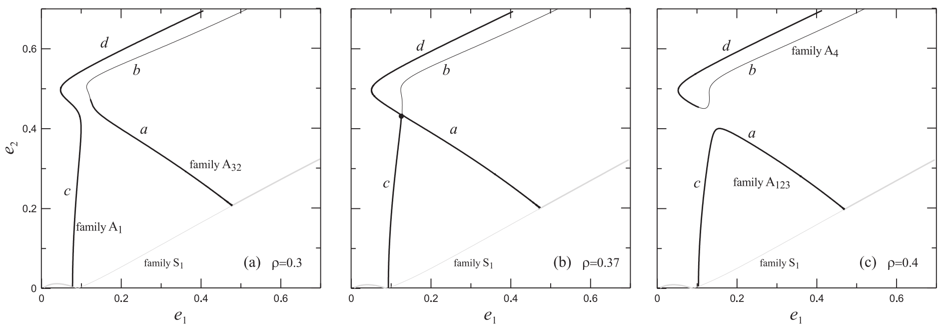

Figure 1: The "collision–bifurcation" of the families A_1 and A_{32} at $\rho \approx 0.37$ and the formation of the new families A_4 and A_{123}.

2. BIFURCATIONS IN 2/1 RESONANCE

Considering the 2/1 external resonance of the restricted problem ($\rho = 0$) we mention the existence of the following families: (i) in the circular problem we have a symmetric family II from which an asymmetric family A_1 bifurcates. (ii) in the elliptic problem we have a symmetric family S_1 and an asymmetric family A_2, which bifurcate from family II (iii)an asymmetric family A_3 which bifurcates from S_1 within the framework of the elliptic problem. In the general problem, for $\rho \approx 0.275$, A_3 and A_2 go through a bifurcation and generate a new family A_{32}.

The evolution of the family A_1, as ρ increases, is regular without structural changes up to $\rho \approx 0.37$. In Fig. 1a, which corresponds to $\rho = 0.3$, it is shown that the family A_1 has come close to the family A_{32}. At $\rho \approx 0.37$ the two families collide and two new families are generated, namely the family A_4 and the family A_{123} (Fig. 1c). We call such a bifurcation as a *collision-bifurcation.* In this case only the family A_{32} has an orbit of critical stability, which separates the family in a stable (a) and in an unstable part (b). The family A_1 is whole stable and there is no clear border between its parts c and d. After the bifurcation, an orbit of critical stability is shown only in the new family A_4, while the family A_{123} is whole stable and starts and ends at bifurcation points of the symmetric family S_1.

Now, we restrict ourselves to the evolution of the family A_{123} for $\rho > \bar{\rho}_2$. As it is shown in Fig. 2a, as ρ increases, the ending points of the family move on along the family S_1 in opposite direction and the family shrinks and, finally disappears at $\rho \approx 1.034$. In Fig. 2b we present the above transition by considering the stability index b_2 (Hadjidemetriou, 2006) for the orbits along the family S_1. In this case only the index b_2 indicates the stability, since it is $-2 < b_1 < 2$. The horizontal axis of the associated plot indicates the eccentricity of the periodic orbits along the family S_1. As ρ increases, the curve of b_2 values is raised continually and for $\rho > 1.034$ is located above the value $b_2 = -2$. Thus, the unstable part of S_1 disappears and, consequently, the family A_{123} disappears too. We call such a bifurcation *break-bifurcation.* After this bifurcation, as $\rho \to \infty$, the family S_1 does not show any structural changes and can be assumed to coincide the family S_1' for the internal 2/1 resonance.

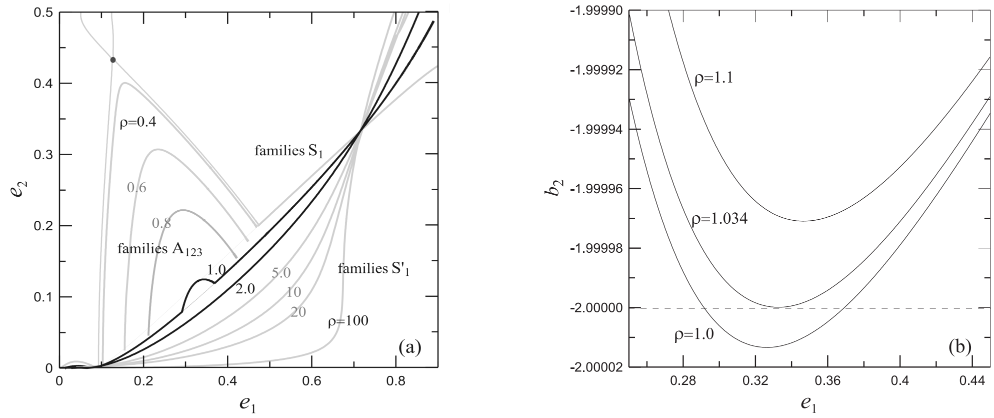

Figure 2: a) The evolution of the families A_{123} and S_1 as ρ passes the critical value $\rho \approx 1.034$ and takes large values. b) The stability index b_2 along the family S_1, which determines the interval of instability ($b_2 < -2$) and the bifurcation points (at $b_2 = -2$) of the family A_{123}.

3. BIFURCATIONS IN 3/1 RESONANCE

In Voyatzis and Hadjidemetriou (2006), it is shown that the 3:1 resonance shows four families S_i, $i = 1, .., 4$ of symmetric periodic orbits which bifurcate from the circular family. Voyatzis (2008) computed these families for a wide range of the mass ratio ρ and verified the results obtained by Michtchenko et al (2006). In the following we consider the evolution of families of asymmetric periodic orbits as ρ increases.

In Fig. 3(left) we present the 3/1 resonant asymmetric family A_4 and A_{43}, which bifurcate from the symmetric family S_4, for various values of ρ. The bifurcation point B_4, as ρ varies, forms the characteristic curve $\mathbf{B}_4$. We can obtain that at the critical value $\rho \approx 0.52$ the characteristic curves show a structural change at the point C. For $\rho < 0.52$ we have the family A_4 which extends up to high values of eccentricities. For $\rho > 0.52$ we have the family A_{43}, which terminates at the bifurcation point B_3 that belongs to the symmetric family S_3.

In Fig. 3(left) there is a region which is not occupied by the families A_4 and A_{43}. This region is indicated by the text "families A_3 and A_0" and contains new families of asymmetric periodic orbits, which, as ρ increases, evolve as it is shown in the panels of Fig. 3(right). We showed above that the bifurcation points B_3 are ends of the families A_{43} for $\rho > 0.51$. For lower values of ρ, the points B_3 are bifurcation points of new families, called families A_3, which exist as $\rho \to 0$. Starting from small values of ρ (e.g. panel (a)), apart from A_3 we obtain the asymmetric family A_{00} whose characteristic curve forms a loop. For $\rho \approx 0.175$ the families A_{00} and A_3 involve in a *collision-bifurcation* at point B_c (panel (b)) but, in this case, the two families join after the bifurcation and form one family, called again A_3 (since it still bifurcates from B_3). For $\rho > 0.52$, family A_3 shows a structural change and now the family is called A_0 and its bifurcation point is computationally undetermined.

Combining the family structures shown in Figs. 3(left) and 3(right), we can obtain that for $\rho < 0.52$ we have the families A_3 and A_4. At $\rho \approx 0.52$ the two families go

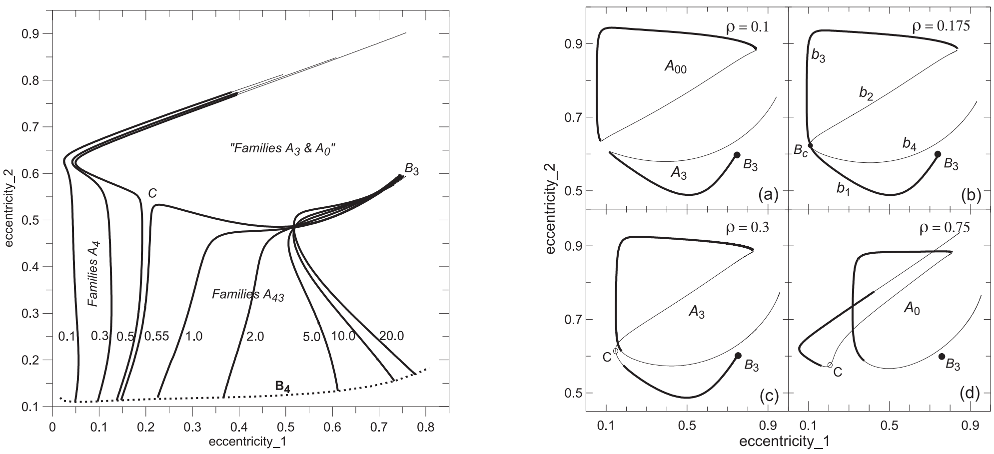

Figure 3: (left) Families A_4 and A_{43} of asymmetric periodic orbits presented in the projection plane $e_1 - e_2$. Bold dotted curves indicate the bifurcation line $\mathbf{B}_4$. The value of the mass ratio ρ is indicated for each curve. (right) Additional 3/1 resonant families. The panels (a)-(d) show the transition from the family A_{00} to A_0 through the involvement of the family A_3.

through a *collision-bifurcation* and for $\rho > 0.52$ we get the families A_{43} and A_0.

4. CONCLUSIONS

In this paper we demonstrated some important bifurcations which takes place in resonances of the three body problem of planetary type. The so called *collision-bifurcations*, take place when, by varying the ratio of the planetary masses, two families collide in the space of initial conditions. After such a collision, new families are generated and the characteristic curves of periodic orbits show substantial changes, which definitely are followed by topological changes of phase space.

Acknowledgment

The authors would like to express their gratitude to Prof. Hadjidemetriou for his supervision and support to their research.

References

Beaugé, C., Ferraz-Mello, S. and Michtchenko, T.: 2003, *Astrophys. J.*, **593**, 1124.
Hadjidemetriou, J. D.: 1975, *Cel. Mech. Dyn. Astr.*, **12**, 155.
Hadjidemetriou, J. D.: 2006, *Cel. Mech. Dyn. Astr.*, **95**, 225.
Michtchenko, T. A., Beaugé, C. and Ferraz-Mello, S.: 2006, *Cel. Mech. Dyn. Astr.*, **94**, 411.
Voyatzis, G. and Hadjidemetriou, J. D.: 2005, *Cel. Mech. Dyn. Astr.*, **93**, 265.
Voyatzis, G. and Hadjidemetriou, J. D.: 2006, *Cel. Mech. Dyn. Astr.*, **95**, 259.
Voyatzis, G., Kotoulas, T. and Hadjidemetriou, J. D.: 2008, submitted.
Voyatzis, G.: 2008, *Astrophys. J.*, **675**, 802.