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QUASI-CRITICAL ORBITS FOR ARTIFICIAL LUNAR SATELLITES

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Abstract. We study the problem of critical inclination orbits for artificial Lunar satellites, under the combined effects of the J_2 and C_{22} terms of the lunar potential and lunar rotation. We show that, at the fixed points of the averaged Hamiltonian, the inclination and the argument of pericenter do not remain constant simultaneously. Instead, there are *quasi-critical* solutions, for which the argument of pericentre librates around a constant value. These solutions represent smooth curves in phase space, which determine the dependence of the quasi-critical inclination on the initial nodal phase. The amplitude of libration of the argument of pericentre would be quite large for a non-rotaning Moon, but it is reduced to $< 0^\circ.1$, when a uniform rotation of the Moon is considered.

1. INTRODUCTION

We consider an artificial satellite in orbit around the Moon. We use a rotating frame whose origin is at the centre of the Moon, the x axis passes through the longest lunar meridian and the $x - y$ plane coincides with the lunar equatorial plane. This frame rotates at the rate of the Moon's synchronous rotation n. The Hamiltonian of the problem has the following form

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{J_2} + \mathcal{H}_{C_{22}} + \mathcal{H}_n \tag{1}$$

$\mathcal{H}_0$ corresponds to the Keplerian term, while $\mathcal{H}_{J_2}$ and $\mathcal{H}_{C_{22}}$ express the perturbing terms. $\mathcal{H}_n$ describes the rotation of the Moon. Using selenographic coordinates, the equation reads

$$\mathcal{H} = \left(\frac{u^2}{2} - \frac{\mu}{r}\right) + \frac{\epsilon\mu}{r^3} P_{20}(\sin\phi) + \frac{\delta\mu}{r^3} P_{22}(\sin\phi)\cos 2\lambda - n p_\phi, \tag{2}$$

where the term $\frac{u^2}{2} - \frac{\mu}{r}$ is the unperturbed problem, (r, ϕ, λ) are the selenographic coordinates, n the angular velocity of the Moon, $\epsilon = J_2 R^2$, $\delta = -C_{22} R^2$, P_{nm} the associated Legendre polynomials and R the mean equatorial radius of the Moon. The angles ϕ, λ can be written as functions of the usual elliptic orbital elements $[a, e, I, \Omega, \omega, M]$.

Translating into the canonical set of Delaunay variables $[l, g, h, L, G, H]$ and averaging over the fast angle, l, the first order averaged Hamiltonian was obtained in [1]

$$\bar{\mathcal{H}} = -\frac{\mu^2}{2L^2} + \frac{\epsilon\mu^4}{4G^3L^3} - \frac{3\epsilon\mu^4H^2}{4G^5L^3} + \frac{3\delta\mu^4\cos 2h}{2G^3L^3} - \frac{3\delta\mu^4H^2\cos 2h}{2G^5L^3} - nH \tag{3}$$

As we have checked, the same result can be obtained using a simpler method, described in [3]. The equations of motion are

$$\dot{l} = \partial\bar{\mathcal{H}}/\partial L \tag{4}$$

$$\dot{L} = -\partial\bar{\mathcal{H}}/\partial l = 0 \tag{5}$$

$$\dot{g} = \partial\bar{\mathcal{H}}/\partial G = -\frac{3\epsilon\mu^4}{4G^4L^3}\left(1 - 5\frac{H^2}{G^2}\right) + \frac{3\delta\mu^4}{2G^4L^3}\cos 2h\left(-3 + 5\frac{H^2}{G^2}\right) \tag{6}$$

$$\dot{G} = -\partial\bar{\mathcal{H}}/\partial g = 0 \tag{7}$$

$$\dot{h} = \partial\bar{\mathcal{H}}/\partial H = -n - \frac{3\epsilon\mu^4H}{2G^5L^3} - \frac{3\delta\mu^4}{G^5L^3}H\cos 2h \tag{8}$$

$$\dot{H} = -\partial\bar{\mathcal{H}}/\partial h = \frac{3\delta\mu^4H}{G^3L^3}\sin 2h - \frac{3\delta\mu^4}{G^5L^3}H^2\sin 2h \tag{9}$$

The averaged system is integrable and separable. Hence, $H(h)$ can be found. Substituting in equation (6) and dividing by (8), we get the corresponding expression for $g'(h)$.

2. ANALYSIS AND RESULTS

The constant value of I, for which the time derivative of g is zero, is called *critical inclination*. This value (if it exists), corresponds to a fixed point in phase space. Apart from such solutions, there can be solutions for which h and H perform small-amplitude oscillations, leading to a constant mean value of the argument of pericentre ($\langle g'(h)\rangle_h = 0$). We define these as *quasi-critical* orbits. These solutions are represented by smooth curves in the phase diagram. Note that, (Fig. 1), all the initial values (h_0, H_0), that give quasi-critical orbits belong to the same trajectory. Thus, there exists in fact a critical value of the action. Calculating $\langle g'(h)\rangle_h$ for $0 \le h \le 90°$ and a range of semimajor axis and eccentricity values ($2,500 \le a \le 4,500$ km, $0 \le e \le 0.2$), we obtain the following results:

In this study, the effect of the Earth was not included. For the values of the semimajor axis used, it is an order of magnitude smaller than the effect of J_2.

The quasi-critical inclination I_{qc} depends on h, but not in the same way described in [1, 2]. The rotation of the Moon plays a major role in this dependence. Specifically, when $n = 0$, $52° \le I_{qc} \le 82°$ for the "$J_2 + C_{22}$" model and $27° \le I_{qc} \le 90°$ for the "C_{22}" model. In the last case, for $40° \le h \le 50°$, there is no I_{qc}, while for $h = 45°$, $\dot{g} = 0$ but $\dot{I} \neq 0$. When $n \neq 0$, $I_{qc} \approx 63°.4$, for the "$J_2 + C_{22} + n$" model (Fig. 2) and $I_{qc} \approx 26°.5$ for the "$C_{22}+n$" model. Changes in the eccentricity have a minimal effect

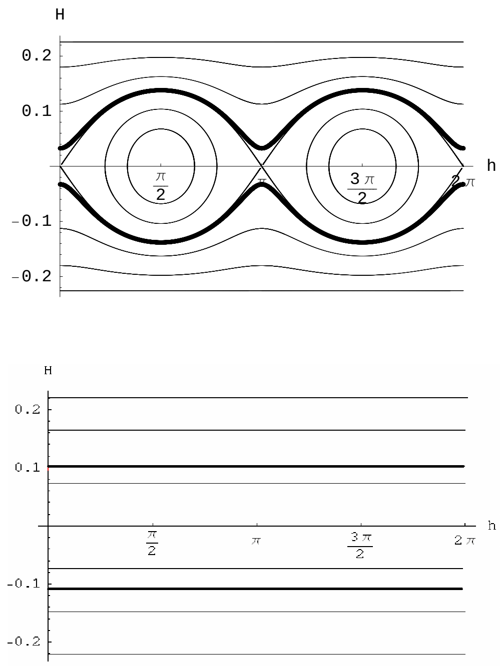

Figure 1: Phase diagrams for the "$J_2 + C_{22}$" (left) and "$J_2 + C_{22} + n$" (right) models. The bold curves represent quasi-critical orbits, while the fixed points are not critical ($\dot{g} \neq 0$).

on the value of the quasi-critical inclination, both in the rotating and non-rotating case, while the libration amplitude of g is an order of magnitude larger when n is not taken into account.

To confirm that the effect of second or higher order short-period terms on I_{qc} is negligible, we ntegrated numerically the equations of motion of both the averaged and the full system. As expected, there are only minimal differences, namely short-period, small-amplitude oscillations of a and e about their mean values and very small differences for I, g and h.

3. CONCLUSION

The effect of the lunar rotation is to minimise the dependence of the quasi-critical inclination on the longitude of the node. When rotation is ignored, a strong dependence on h arises for the C_{22}/J_2 ratio of the Moon. With or without rotation, the fact that the averaged inclination librates, does not allow us to speak about critical values in the strict sense. There exists however a critical value of the corresponding *action* variable.

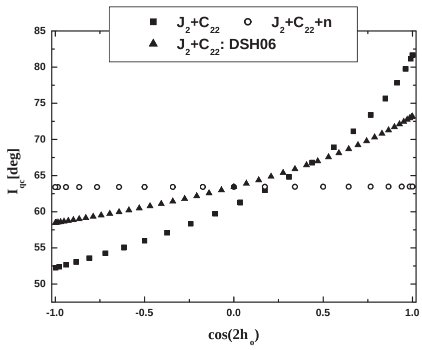

Figure 2: Quasi-critical inclination as a function of $\cos(2h)$ (DSH06 means the solution of De Saedeleer and Henrard 2006)

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