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CLOSE ENCOUNTERS IN THE CALEDONIAN SYMMETRIC FOUR-BODY PROBLEM

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Abstract. The Caledonian Astro-dynamics Research Group has developed the Caledonian Symmetric Four-Body Problem (CSFBP) which is a restricted four-body system with a symmetrically reduced phase space. The main limitation of the CSFBP model is that the equations of motion of the problem contain singularities which cause numerical integration algorithms to fail as the system approaches a close encounter. In order to study the close encounters and collision events occurring in the CSFBP, we propose a new global regularization scheme for the CSFBP. The resulting equations of motion can be efficiently integrated by any higher order integrator. The effectiveness of this approach is illustrated for a set of CSFBP orbits.

1. INTRODUCTION

The four-body problem has been studied with the help of geometrical restriction methods. Steves and Roy (1998) used symmetric boundary conditions, in reducing the mathematical complexity of the problem, and introduced the Caledonian Symmetrical Four-Body Problem(CSFBP), which is a restricted four-body system with a symmetrically reduced phase space. The CSFBP proposed by the authors is relevant in studying the stability and evolution of symmetric quadruple stellar clusters and exoplanetary systems of two planets orbiting a binary pair of stars (Steves and Roy 1998, Roy and Steves 2000, Steves and Roy 2001, Széll et al. 2004, 2004a, 2004b). One of the main difficulties in the study of the CSFBP model is the existence of collision singularities which cause numerical integration algorithms to fail as the system approaches a close encounter. Regularization is an efficient tool for numerically integrating dynamical systems with singularities by transforming their corresponding equations of motion into regular ones. The CSFBP needs to be regularized in order to study the close encounters and collision events occurring in the system.

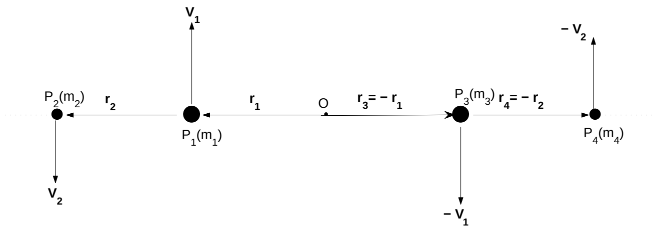

Figure 1: The initial configuration of the coplanar CSFBP.

2. AN OVERVIEW OF CLOSE ENCOUNTERS IN THE CSFBP

The coplanar CSFBP involves two pairs of distinct masses moving in coplanar, initially circular orbits, starting in a collinear arrangement. Fig. 1 shows the initial configuration of the system. Denoting the four bodies P_1, P_2, P_3, P_4, then P_1 and P_3 have mass M and are symmetrically placed on opposite sides of the centre of mass of the system, while P_2 and P_4 have mass m and are also symmetrical. Thus $m_1 = m_3 = M, \quad m_2 = m_4 = \mu M$, where the mass ratio $\mu = m_2/m_1$. The radius and velocity vectors of the bodies with respect to the centre of mass of the four-body system are given by $\mathbf{r}_i$ and $\dot{\mathbf{r}}_i$, respectively, where $i = 1, 2, 3, 4$. Steves and Roy (1998) exploited both past-future and dynamical symmetries and reduced the number of variables of the system to the study of the dynamical behavior of one of the binary pairs, the other binary pair's motion being a mirror image of the first binary pair's motion. Thus according to the symmetry conditions,

$$\mathbf{r}_1 = -\mathbf{r}_3, \qquad \mathbf{r}_2 = -\mathbf{r}_4$$
$$\mathbf{V}_1 = \dot{\mathbf{r}}_1 = -\dot{\mathbf{r}}_3, \qquad \mathbf{V}_2 = \dot{\mathbf{r}}_2 = -\dot{\mathbf{r}}_4. \tag{1}$$

In the presence of close encounters, the accuracy of numerical integrations of the original equations of motion drops near these singularities. Szell, Steves and Erdi (2004) have identified four types of two-body collisions in the CSFBP; "12" type double binary collision (collisions occurring in the binary formed between P_1 and P_2 and the symmetrical binary formed between P_3 and P_4), "14" type double binary collision (collisions occurring in the binaries formed between P_1 and P_4 and between P_2 and P_3) and "13" type (binary formed between P_1 and P_3) and "24" type (binary formed between P_2 and P_4) single binary collisions. There are singular points associated with each of these collisions. In the next section, we will present a global regularization method which can handle these two-body collisions in the system. (Note that in general orbits cannot be integrated beyond four-body collisions, which occur when all of these binary collisions occur simultaneously.)

3. THE REGULARIZATION PROCEDURE

The regularization scheme presented in this paper makes use of a Levi-Civita type coordinate transformation and a time transformation function similar to the Aarseth and Zare scheme (Aarseth and Zare 1974). An approach of this kind has been previously used for the simpler collinear symmetrical four-body problem where only "12" and "24" binary collisions can occur (Sweatman 2002, 2006). We develop the method further by modifying the regularization algorithm applied by Heggie (1974) and Heggie and Sweatman (1991) to incorporate the symmetries of the CSFBP model. We use the Hamiltonian formalism to maintain the canonicity of transformations.

Let $\omega_1, \omega_2, \omega_3$ and ω_4 be the magnitudes of the momenta of the four bodies P_1, P_2, P_3 and P_4, respectively. Assume that the center of mass is initially at rest at the origin. Then the Hamiltonian for the CSFBP system can be written in the form

$$H = \frac{1}{M}(\omega_1^2 + \omega_2^2) + \frac{1}{\mu M}(\omega_3^2 + \omega_4^2) - 2\mu M^2 \left(\frac{1}{r_{12}} + \frac{1}{r_{14}}\right) - \frac{M^2}{r_{13}} - \frac{\mu^2 M^2}{r_{24}}, \qquad (2)$$

where the inter body distances r_{12}, r_{14}, r_{13} and r_{24} are equal to zero at the singular points which represent the physical collisions discussed in Section 2. In order to regularize these singularities, we map the physical plane into a parametric plane (Q_i, P_i) using a series of transformation equations so that the transformed Hamiltonian function has no singularities near two-body close encounters. The first step in our regularization procedure is to introduce a set of coordinates q_j (j=1 to 8), which are the x and y components of the inter-body vectors.

Then on each inter-body vector, we perform a Levi-Civita transformation of the form

$$q_k + iq_{k+1} = f + ig = (Q_k + iQ_{k+1})^2. \qquad (3)$$

where $i = \sqrt{-1}$, (q_k, q_{k+1}) refers to a physical plane and (Q_k, Q_{k+1}) refers to a parametric plane, k= 1,3,5,7 and f and g are two conjugate harmonic functions which satisfy the Cauchy-Riemann relations. The physical momenta will be transformed to a set of conjugate momenta.

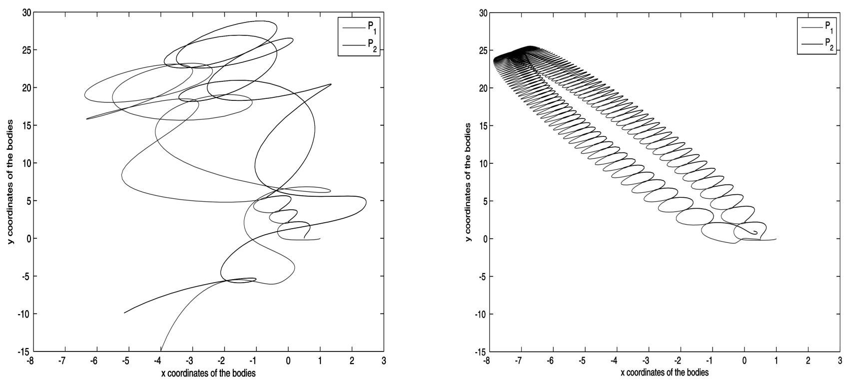

Figure 2: An illustration of a physical orbit of bodies P_1 and P_2 calculated for 200 time steps from the same initial conditions using (a) non-regularized equations of motion, (b) regularized equations.

In the next step, we introduce a fictitious time τ, which is a key factor for the regularizing effect. We can find a variety of choices for the time transformation function in the literature. However in the case of the CSFBP, a time re-scaling factor similar to the Aarseth-Zare scheme was found to be very effective in regularizing the close encounters.

$$\begin{aligned} \frac{dt}{d\tau} &= g = \frac{r_{12}r_{23}r_{13}r_{24}}{(r_{12}+r_{23}+r_{13}+r_{24})^{5/2}} \qquad (4) \\ &= \frac{(Q_1^2+Q_2^2)(Q_3^2+Q_4^2)(Q_5^2+Q_6^2)(Q_7^2+Q_8^2)}{(Q_1^2+Q_2^2+Q_3^2+Q_4^2+Q_5^2+Q_6^2+Q_7^2+Q_8^2)^{5/2}}. \end{aligned}$$

The Hamiltonian in the regularized phase space becomes

$$\Gamma = g(H - h_0), \qquad (5)$$

where h_0 is the constant value of energy. From this, we can obtain canonical equations of motion without any singularities.

We have tested the new integration scheme for the CSFBP on a variety of problems. Fig. (2) shows the results of a particular system in which we start with P_1 and P_2 in an eccentric binary with frequent close encounters. The numerical integration has been performed from the same initial conditions with (a) the equations of motion derived from the original (non-regularized) Hamiltonian equation (2) (Fig. 2a), and (b) those of the regularized Hamiltonian equation (5) (Fig. 2b). The orbits are similar for a small time period (approximately three periods of the binary). The non-regularized orbit then becomes chaotic as the collision events occur and the associated energy error increases with respect to time. The regularized orbit remains regular and well-behaved for large time durations. Our numerical experiments have shown that the regularized system of equations have excellent energy-conserving capability and the energy error remains stable and minimal.

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