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HOMOCLINIC CONNECTIONS IN THE HILL PROBLEM WITH RADIATION

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Abstract. Asymptotic motion around the collinear equilibrium points of the Hill problem is considered when the primary is radiating. Specifically, asymptotic orbits to the collinear equilibrium points and to Lyapunov orbits are determined. Transversality of the latter is achieved by construction of appropriate surface of section portraits of the unstable manifolds.

1. INTRODUCTION

Motion around the collinear equilibrium points of the restricted three–body problem or its Hill limiting case has attracted considerable attention in the last decades (e.g. Simó and Stuchi, 2000; and references therein). An interesting case of such motion is the case when the moving particle traces orbits which asymptotically emanate/terminate from/at these points or the Lyapunov periodic orbits around them (see for example Llibre, et al., 1985; Gómez and Mondelo, 2001; Markellos et al., 2003).

Asymptotic orbits at collinear equilibrium points may be considered as limiting cases of asymptotic orbits to the Lyapunov periodic orbits, i.e. they are orbits emanating/terminating from/at the collinear equilibria themselves, instead of from finite orbits around them. Therefore, an asymptotic orbit at a collinear equilibrium can be used as a reference orbit since its existence indicates the existence, in its immediate neighbourhood, of an infinity of orbits asymptotic to the Lyapunov periodic orbits (Kalantonis et al, 2006). These orbits have been studied by Deprit and Henrard (1965) and Perdios and Markellos (1990), in the framework of the restricted three–body problem. On the other hand, orbits asymptotic to Lyapunov orbits are important from a theoretical and practical point of view since they cause the destruction of invariant tori, while they can also be used for the design of trajectories for space missions (Koon et al., 2000; Gómez et al., 2005).

In this paper we study homoclinic orbits at the collinear equilibrium points as well as at the Lyapunov orbits, of a variant of Hill's problem in which the primary ("the Sun") is a source of radiation. To determine orbits which asymptotically terminate at these points we use fourth order expansions with respect to a small orbital parameter. For the determination of asymptotic orbits to the Lyapunov periodic orbits we compute the corresponding unstable manifolds. A number of appropriate surface of section portraits are constructed in order to detect transversality of the stable and unstable manifolds. Several such orbits have been determined for the specific value of the radiation factor $Q_1 = 0.5$.

In rotating coordinates, the Hill problem, in which the larger primary is a source of radiation, is described by the following equations of motion (Markellos et al. 2000):

$$\ddot{x} - 2\dot{y} = \frac{\partial W}{\partial x}, \qquad \ddot{y} + 2\dot{x} = \frac{\partial W}{\partial y}, \qquad W = \frac{3x^2}{2} - Q_1 x + \frac{1}{r}, \tag{1}$$

where W is the potential function and $r = \sqrt{x^2 + y^2}$. The equation of the Jacobi integral is $2W - (\dot{x}^2 + \dot{y}^2) = \Gamma$, where Γ is the Jacobi constant. These equations result from the corresponding variant of the restricted three–body problem where the larger primary is a source of radiation by placing the origin at the smaller primary, appropriate rescaling of lengths and radiation factor.

The problem admits two unstable collinear equilibrium points; L_1 on the negative axis and L_2 on the positive axis. The $x-$axis is an axis of symmetry but, contrary to the classical Hill problem, the $y-$axis is not. The positions of these equilibria can be found analytically (Markellos et al., 2000).

2. HOMOCLINIC ORBITS AT COLLINEAR EQUILIBRIA

We express the solution near an equilibrium point L_i, $i = 1, 2$, to fourth order terms in a small parameter ϵ:

$$\xi(t) = \sum_{j=1}^{4} \epsilon^j \phi_j(t), \qquad \eta(t) = \sum_{j=1}^{4} \epsilon^j \omega_j(t), \tag{2}$$

where ξ, η are the coordinates with origin L_i, $i = 1, 2$. To determine an asymptotic orbit to L_i, $i = 1, 2$, we consider the solutions which correspond to the real eigenvalues. Two outgoing and two incoming solutions exist. Here we consider only the asymptotic solutions which remain in the sphere of influence of the secondary. The solution to first–order, corresponding to the eigenvalue $\lambda_0 > 0$, is directly obtained in the form $\xi(t) = \epsilon e^{\lambda_0 t}$, $\eta(t) = \epsilon g_1 e^{\lambda_0 t}$, where:

$$g_1 = \frac{\lambda_0^2 - A_1}{2\lambda_0} = \frac{2\lambda_0}{B_1 - \lambda_0^2}, \qquad \text{with} \qquad A_1 = 3 + \frac{2}{r_0^3}, \qquad B_1 = -\frac{1}{r_0^3}. \tag{3}$$

The corresponding fourth order solution is of the form:

$$\xi(t) \; = \; \sum_{j=1}^{4} \epsilon^j h_j e^{j\lambda_0 t}, \qquad \eta(t) \; = \; \sum_{j=1}^{4} \epsilon^j g_j e^{j\lambda_0 t}, \tag{4}$$

where the coefficients h_j, g_j, $j = 2, 3, 4$, are determined by successive approximations. Due to the symmetry of the problem w.r.t. the $x-$axis we are able to determine the initial conditions of the incoming asymptotic orbit from those of the outgoing orbit by changing the signs of y_0 and $\dot{x}_0$.

Due to the same symmetry, transversality of the unstable with the stable manifold of the equilibrium point is detected when the orbit reaches this axis peprpendicularly,

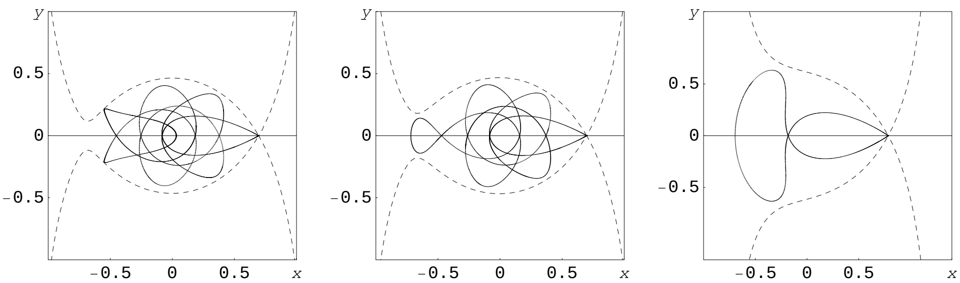

Figure 1: Homoclinic orbits at collinear equilibrium point L_2.

i.e. with $\dot{x}(Q_1) = 0$. Therefore, we scan the Q_1-axis and for each value of Q_1 integrate the equations of motion, using appropriate initial conditions from our analysis, up to the $n-$th crossing of the orbit with the $x-$axis. The roots of the function $\dot{x}(Q_1)$ will indicate the existence of homoclinic orbits at the collinear equilibrium point. When a homoclinic orbit has been detected, we can determine it accurately by applying well–known differential corrections procedures or by a more refined scanning. Numerical data of many homoclinic orbits are available for the readers. In Fig. 1 we show homoclinic orbits (continuous lines) at the positive collinear equilibrium point for the values of the radiation factor $Q_1 = 0.01471899$, $Q_1 = 0.03551808$ and $Q_1 = 0.72629021$ (from left to right) together with the corresponding zero– velocity curves (dashed lines).

3. HOMOCLINIC ORBITS TO LYAPUNOV ORBITS

In order to detect homoclinic orbits to periodic orbits of the Lyapunov family we shall use again the symmetry property of the problem w.r.t. the $x-$axis. For a particular Lyapunov orbit the initial four–dimensional phase space is reduced to a three–dimensional subspace of iso–energetic orbits due to the equation of the Jacobi integral, for the specific value of the Jacobi constant of the Lyapunov orbit. The final reduction of the phase space to a two– dimensional sub–space is carried out by considering the cuts of the unstable manifold of the Lyapunov orbit with the $x-$axis. If the unstable manifold of the Lyapunov orbit has a perpendicular intersection with the $x-$axis, i.e. the horizontal component of the velocity is zero ($\dot{x} = 0$), then transversality is achieved and a homoclinic orbit to the Lyapunov orbit exists. The construction of the corresponding stable–unstable manifolds is based on a linear analysis (Símo and Stuchi, 2000).

In Fig. 2(a) the phase portraits, in the $(x, \dot{x})$ plane, of the unstable manifolds (first cuts with the $x-$axis) for various Lyapunov orbits are presented for the value of the radiation factor $Q_1 = 0.5$. Each curve corresponds to a specific periodic orbit and the points of the curve located on the $x-$axis, i.e. with $\dot{x} = 0$, denote the existence of homoclinic orbits. The tangential curve (dashed line) marks the onset of transversality. Note that the "center" of the innermost elliptic curve represents the limiting case of the unstable manifolds of the Lyapunov orbits, i.e. the one–dimensional unstable manifold of the equilibrium point L_1. In Figs. 2(b) and (c) we show the phase portraits of the second and third cuts of the unstable manifolds

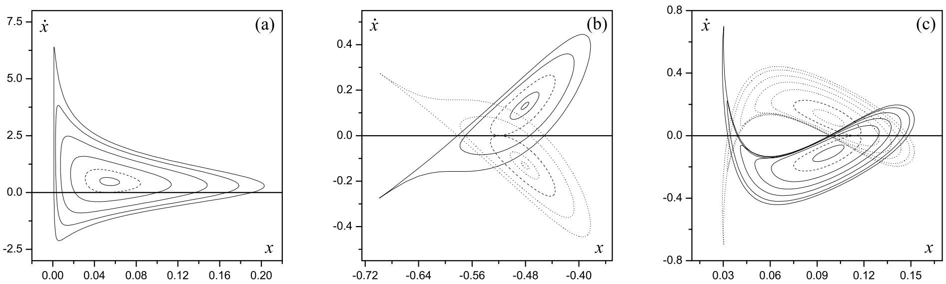

Figure 2: The phase portraits of the unstable manifolds of Lyapunov families around L_1 for the (a) $n = 1$ cuts, (b) $n = 2$ cuts and (c) $n = 3$ cuts.

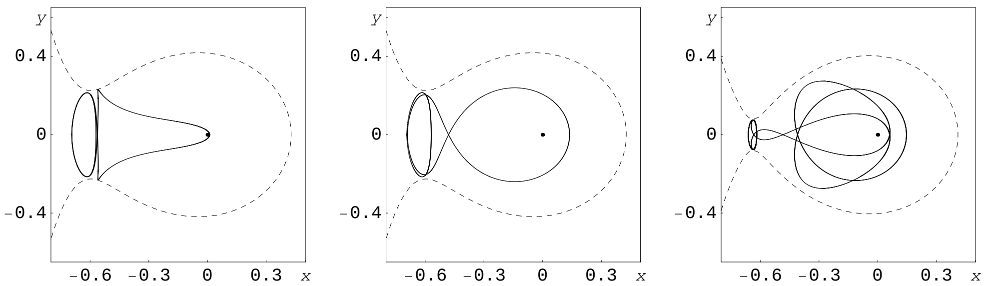

Figure 3: Homoclinic orbits to Lyapunov orbits around L_1.

for various Lyapunov orbits together with the portraits of the corresponding stable manifolds (dotted lines). In these two cases we observe that transversality of the stable with the unstable manifolds occurs also outside the $x-$axis indicating the existence of non– symmetric homoclinic orbits. In Fig. 3 some symmetric homoclinic orbits to Lyapunov periodic orbits are shown.

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