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FULFILLMENT OF THE CONDITIONS FOR APPLICATION OF NEKHOROSHEV THEOREM IN THE ASTEROID BELT

R. PAVLOVIĆ and Z. KNEŽEVIĆ
Astronomical Observatory, Volgina 7, 11060 Belgrade 38, Serbia
E–mail: rpavlovic@aob.rs
E–mail: zoran@aob.rs

Abstract. A semianalytical procedure to check the fulfillment of the conditions for application of Nekhoroshev's theorem has been recently developed and applied to the non resonant domain of a real dynamical system (Pavlović and Guzzo, 2008, MNRAS **384**, 1575). Here the analogous procedure has been applied to the asteroids, members of Koronis and Veritas families. Following the procedure to compute secular resonant proper elements, we used an extended Hamiltonian that includes terms linear in planetary eccentricities and inclinations. By suitable transformations the Hamiltonian is expressed in an appropriate form, and the conditions of convexity, quasi–convexity and 3-jet nondegeneracy checked.

1. INTRODUCTION

The Nekhoroshev theorem provides an exponential estimate of the stability time for quasi-integrable non-degenerate Hamiltonian systems. Nekhoroshev (1977) proved this stability result for the Hamiltonian functions that are analytic perturbations of steep functions. To practically apply the theorem it is thus necessary to verify that the conditions for application of the Nekhoroshev theorem are fulfilled in the regions of the phase space of interest (in our case, in the main asteroid belt). The first attempt to check the fulfillment of these conditions has been recently made by Pavlović and Guzzo (2008) who used a simple integrable approximation of the asteroid Hamiltonian, the so-called *Kozai's Hamiltonian*, to show that it is indeed steep and that the conditions of Nekhoroshev theorem are in this case fulfilled for most of the selected asteroids in the regions of the asteroid families Koronis and Veritas. The application of the theorem to the real asteroids, such as that by Guzzo et al. (2002), has thus been justified.

The main problem with the approach of Pavlović and Guzzo (2008) is that they have checked the fulfillment of the conditions for application of the Nekhoroshev theorem by using the Hamiltonian which is known to be only a rough approximation of the real dynamics. Kozai's (1962) approximation is obtained by averaging the full N-body Hamiltonian over the low-order resonant terms, by subsequent expansion of the averaged Hamiltonian in Taylor series with respect to the eccentricities and inclinations of the perturbing planets, and by keeping only the leading terms of these procedures. Thus, it lacks terms of higher degrees and orders, which in many cases can be more important than those accounted for. In the present paper we tried to partly overcome this problem by introducing additional terms in the Hamiltonian, keeping it still integrable and simple enough to allow for the cumbersome computations involved with the checking of conditions. We closely followed, up to the point, the procedure by Morbidelli (1993) to compute asteroid resonant proper elements, thus making use of Kozai's terms complemented by the terms linear in planetary eccentricities and inclinations.

It is well known that the condition of steepness involved with the proof of the Nekhoroshev theorem is originally written in an implicit form difficult to use in practical applications. Instead, however, one can use some stronger sufficient conditions, which can be explicitly written in terms of the derivatives of the Hamiltonian of the problem. These conditions are convexity, quasi-convexity and 3-jet (Nekhoroshev, 1977), which were used, for example, by Benettin et al. (1998) to study the long-term stability of the Lagrangian points.

The definition of the conditions and the description of the procedure of their computation are given in Pavlović and Guzzo (2008). In brief, once the integrable Hamiltonian is known and the corresponding Hessian derived, one proceeds with the check of the signs of its eigen values. If all the three signs are the same (more precisely, if the Hessian is either positive or negative definite), the Hamiltonian is *convex*. If one of the signs differ, the restriction of the Hessian to the plane orthogonal to the frequency vector ω is considered; if it is either positive or negative definite, that is, if the signs of its two eigen values are the same, the Hamiltonian is *quasi-convex*. If the signs differ one must first compute vectors $u^{\pm}$ in the plane orthogonal to the frequency vector ω defined by an additional condition, and then check the fulfillment of the *3-jet* condition itself. This involves computation of the third derivatives of the Hamiltonian, which in the case of an asteroid Hamiltonian must be done by means of the semi-numerical techniques, extending the techniques introduced by Henrard (1990) and Lemaitre and Morbidelli (1994).

If none of the above conditions are fulfilled, we conclude that the Nekhoroshev theorem cannot be applied for a given asteroid. Since in practice it is difficult to get an exact zero as the result of the above described complex procedure, we used as the threshold value to distinguish between fulfillment of the 3-jet condition and no condition at all, an arbitrary small value of 0.01 (the value used by Lemaitre and Morbidelli 1994, as a resonance boundary criterion). Obviously, increasing this value would increase also the number of "failures", that is of asteroids for which conditions are not fulfilled, and vice versa.

2. HAMILTONIAN AND ITS DERIVATIVES

As already mentioned, Pavlović and Guzzo (2008) checked the conditions for application of Nekhoroshev theorem by using the Hamiltonian derived under the assumtion that all the perturbing planets have circular, planar orbits (Kozai 1962). Here we added to this simple Hamiltonian, as its perturbation, the most important terms for Jupiter and Saturn, linear in e' and i'. The resulting Hamiltonian h can be represented as a sum of the integrable part $h_0 + \mathcal{K}_0$ and of the perturbation $\mathcal{K}_1$.

$$\begin{aligned}\mathcal{K}_1 \quad = \quad & \sum_{k=-n}^{n} \mathcal{K}^1_{1,k}(\Lambda, J, Z) \cos(2k\psi - z - g_5 t - \lambda^0_5) + \\ & \sum_{k=-n}^{n} \mathcal{K}^2_{1,k}(\Lambda, J, Z) \cos(2k\psi - z - g_6 t - \lambda^0_6) + \\ & \sum_{k=-n}^{n} \mathcal{K}^3_{1,k}(\Lambda, J, Z) \cos((2k+1)\psi - z - s_6 t - \mu^0_6). \qquad (1)\end{aligned}$$

Note that the perturbation depends not only on the argument of perihelion and longitude of node of an asteroid, but also on the time through the perturbing planets variables.

To check the fulfillment of the conditions for this extended Hamiltonian we have to get rid of these dependances of the angles and of the time. This is done in two steps: i) by applying Henrard's semi-numerical method we transform $\mathcal{K}_0$ into a Hamiltonian that does not depend on angles (λ, ψ, z), but only on actions (Λ, J, Z) (Henrard 1990; Lemaitre and Morbidelli 1994; Pavlović and Guzzo 2008); note that in this step $\mathcal{K}_1$ still depends on all action-angle variables; ii) by eliminating angles and the time from $\mathcal{K}_1$ using Lie's algorithm of the first order. This means that we are looking for a generating function W_1 as a solution of a homological equation $\{W_1, \mathcal{K}_0\} + \mathcal{K}_1 = 0$, where $\{.,.\}$ denotes Poisson's brackets. Since, following Morbidelli (1993), we made use of K_1 in terms of its Fourier expansion, the explicit form of W_1 is straightforwardly obtained. Relations between the old and new action-angle variables are given by the implicit relations which are solved iteratively (Lemaitre and Morbidelli 1994).

The derivatives up to the third order which appear in the above transformations are given by cumbersome relations (see Pavlović and Knežević 2008). Let us just note that for their computation it is necessary to compute simultaneously the variational equations.

3. RESULTS

The procedure outlined in the previous Section has been applied to a number of members of Koronis and Veritas asteroid families. The obtained results were compared with those by Pavlović and Guzzo (2008) and summarized in Table 1. For Koronis family we found that out of 96 tested members 4 passed from the group "convex" to the group "quasi-convex", 6 from the group "quasi-convex" to group "3-jet", while 3 objects from "3-jet" group now do not fulfill any of the conditions. Thus, only 13 objects in total, or 13% of the sample has changed the class, and none of the objects passed to the group which fulfills the most strict condition of convexity.

For the Veritas family, out of 48 tested objects 2 passed from the group "convex" to group "quasi-convex" and 1 vice versa, 3 passed from "quasi-convex" to "3-jet"; 2 members which previously did not fulfill any of the conditions now fulfill "3-jet" one, while 1 object of the previous "3-jet" group now does not fulfill any of the conditions. Hence, 9 members of the Veritas family, or some 19% of the sample have changed their class.

Table 1: Fulfillment of the conditions for members of Koronis and Veritas families. N is the number of analyzed objects, ΔN gives the number of bodies from the group x which passed to group y ($x \rightarrow y$) when the Hamiltonian $h = h_0 + \varepsilon\mathcal{K}_0$ is extended to become $h = h_0 + \varepsilon\mathcal{K}_0 + \varepsilon\mathcal{K}_1$.

		Koronis		Veritas	
Condition	Label	N	ΔN	N	ΔN
Convexity	1	24	$(1 \rightarrow 2)$ 4	12	$(1 \rightarrow 2)$ 2
Quasi-convexity	2	24	$(2 \rightarrow 3)$ 6	12	$(2 \rightarrow 1)$ 1
					$(2 \rightarrow 3)$ 3
3-jet	3	24	$(3 \rightarrow 0)$ 3	12	$(3 \rightarrow 0)$ 1
None	0	24	$(0 \rightarrow 3)$ 2	12	$(0 \rightarrow 3)$ 2
Summary		96		48	

4. CONCLUSIONS

The results obtained for the families Koronis and Veritas show a comparatively small changes of the previous findings by Pavlović and Guzzo (2008). Only a small fraction of the members shifts from fulfilling of one condition to fulfilling of another one. This was expected as both families are located far from the linear secular resonances introduced in the dynamical model through K_1. However, our primary aim here was to verify the procedure and to demonstrate that the complex algorithm and complicated code are correct and provide meaningful results. We consider that on the basis of the presented results we can safely claim that this goal has been met.

In the forthcoming paper (Pavlović and Knežević 2008) we intend to check the fulfilment of conditions for objects for which we expect more significant changes, that is for asteroids close to the main secular resonances ν_5, ν_6 i ν_{16}. We also plan to further extend the Hamiltonian to include additional terms and to analyze objects in different dynamical regimes.

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