

ASYMMETRIC PERIODIC SOLUTIONS IN THE RESTRICTED THREE-BODY PROBLEM

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Abstract. A systematic numerical exploration of the families of asymmetric periodic orbits of the restricted three-body problem when the primary bodies are equal, is presented. Decades families of asymmetric periodic solutions were found and three of the simplest ones are illustrated. All of these families consist of periodic orbits which are asymmetric with respect to x -axis while are simple symmetric periodic orbits with respect to y -axis (i.e. the orbit has only one perpendicular intersection at half period with y -axis). The three simplest asymmetric families consist, mainly, of unstable periodic solutions but there exist very small, with respect to x , intervals where these families have stable periodic orbits.

1. INTRODUCTION

The study the determination and the calculation of families of asymmetric periodic orbits in the restricted three-body problem, up to now, is associated mainly, with the families of symmetric periodic orbits or with the families which emanate from the triangular equilibrium points of the problem (e.g. Kristiansson, 1933, Strömberg, 1935, Message, 1958, Danby, 1967, Deprit et al., 1967, Markellos et al., 1978, Taylor, 1983, Papadakis, 2008 etc). So, the discovery of asymmetric periodic solutions has been based primarily on bifurcations from families of symmetric orbits at critical points (Hénon, 1965) or studding the asymptotic homoclinic and heteroclinic orbits associated with the triangular equilibrium points L_4 and L_5 using the invariant stable and unstable manifolds (Papadakis, 2006, etc). A systematic exploration of the families of asymmetric periodic orbits is not as simple as in the case of symmetric ones, because now two initial values must be simultaneously adjusted instead of one.

The main goal of the present work is the systematic numerical determination of families of asymmetric periodic orbits which are not associated with the families of symmetric periodic orbits. When we mention asymmetric periodic orbit we mean the periodic solution of the problem which is non-symmetric with respect to the horizontal x -axis. The families were found consist of asymmetric periodic orbits which are non-symmetric with respect to the horizontal x -axis but they are simple-symmetric with respect to the vertical y -axis (i.e. these orbits have only one perpendicular intersection at half period with y -axis). This symmetry with the vertical axis gives

us the possibility for a systematic exploration of families of asymmetric solutions of the problem. The equations of motion of the infinitesimal mass m_3 , of the planar, circular restricted problem of three bodies, in the usual dimensionless rectangular rotating coordinate system are written as (Szebehely, 1967),

$$\ddot{x} - 2\dot{y} = \frac{\partial \Omega}{\partial x}, \quad \ddot{y} + 2\dot{x} = \frac{\partial \Omega}{\partial y} \quad \text{with} \quad \Omega = \frac{1}{2}(x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} \quad (1)$$

and $r_1^2 = (x + \mu)^2 + y^2$, $r_2^2 = (x + \mu - 1)^2 + y^2$, while μ is the mass parameter of the problem. The primaries m_1 and m_2 have masses $1 - \mu$ and μ , and coordinates $(-\mu, 0)$, $(1 - \mu, 0)$ correspondingly. The energy (Jacobi) integral of this problem, is given by the expression $\dot{x}^2 + \dot{y}^2 = 2\Omega - C$ where C is the Jacobi constant.

2. NUMERICAL EXPLORATION OF ASYMMETRIC PERIODIC ORBITS

In the case of a periodic orbit which is asymmetric with respect to the horizontal x -axis, the intersections of the orbit with the x -axis are not perpendicular. There are, however, asymmetric periodic orbits which are symmetric with respect to the vertical y -axis and therefore, in this case, the orbit has perpendicular intersection with the y -axis. These asymmetric solutions can be calculated systematically in the (y, C) plane.

Searching this plane we calculate the network of the families of simple-symmetric periodic orbits with respect to y -axis, i.e. the periodic solutions which have only one perpendicular intersection with y -axis at half period. In Fig. 1 (left) we present this

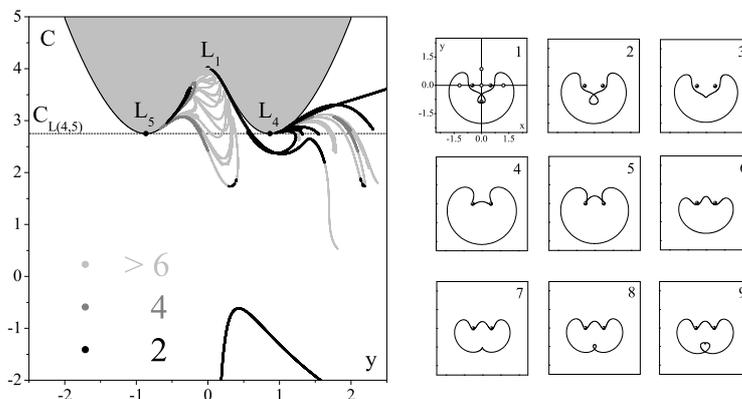


Figure 1: Left: The network of the families of simple symmetric, with respect to y -axis, periodic orbits of the restricted three-body problem with equal primaries. Gray scale indicates the number of the intersections of the periodic orbits with x -axis. Right: The evolution of the asymmetric family asym1. Dots are the equal primaries while the small circles (in the first frame) indicate the positions of the five equilibrium points of the problem.

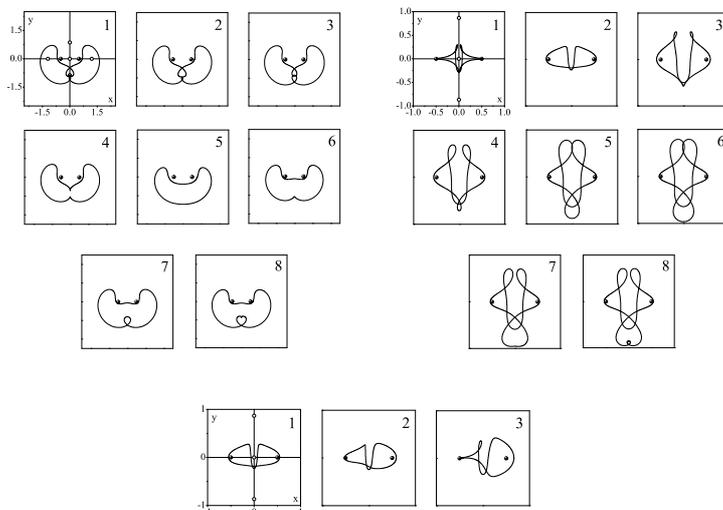


Figure 2: The evolution of the asymmetric families *asym2* (up-left) and *asym3* (up-right). Down: The evolution of the asymmetric family *asym3b* which bifurcates from the asymmetric family *asym3*.

network of the families of the restricted three-body problem when the primary bodies are equal.

The energy integral of the problem is given in introduction and therefore the curves of zero velocity of the problem are defined by eq. $2\Omega - C = 0$, while the regions of possible motions of the third particle are defined by the inequality $2\Omega - C > 0$. In Fig. 1 (left) the forbidden regions are shaded and the positions of the equilibrium points L_i , $i = 1, 4, 5$ are marked. It is obvious that all these families presented in Fig. 1 (left), are not all families of asymmetric periodic orbits. Some of them are the known families of simple-symmetric periodic orbits which have the y -axis as an axis of symmetry (namely the families *l*, *m*, *c*, *k* and *r*, see Hénon, 1965 for details for these families) and some of these characteristic curves correspond to families with higher-multiplicity symmetric periodic orbits. However, the majority of these characteristic curves correspond to families of asymmetric periodic orbits. Some of them are the known families of asymmetric periodic orbits which intersect families of symmetric periodic solutions, namely there are the asymmetric families which bifurcate from the symmetric families *l*, *c* and *r* (see Papadakis, 2008 for details for these asymmetric families). All the other characteristic curves correspond to new families of asymmetric periodic solutions of the restricted three-body problem. For a better understanding of the network of the families in Fig. 1 (left), we use gray scale to indicate the number of the intersections of the periodic orbits with respect to x -axis. So, the black characteristic curves correspond to families of periodic orbits with 2 intersections with x -axis, i.e. these orbits have 2 intersections with y -axis and 2 with x -axis. The gray characteristic curves correspond to families of periodic orbits with 4 intersections with x -axis, i.e. these orbits have 2 intersections with y -axis and 4 with x -axis and so on. We calculated decades of families of asymmetric periodic orbits using the

initial conditions of the (y, C) plane and we present three of the simplest one in Figs. 1 and 2. The first family, say *asym1*, consists of asymmetric periodic orbits close to primaries and its evolution is presented in Fig. 1 (right). This family starts (frame 1) and ends (frame 9) with homoclinic asymptotic orbit on the triangular equilibrium point L_5 . We note here that in this figure dots are the equal primaries while the small circles (in the first frame) indicate the positions of the five equilibrium points of the problem.

The second family, say *asym2*, similarly as the previous one, has as termination members homoclinic orbits on the triangular equilibrium point L_5 (Fig. 2, up-left). Isolated periodic orbits of these two families (*asym1*, *asym2*) have been given by Strömberg (1930), Danby (1967) and Papadakis (2006). The evolution of the third family, say *asym3*, is presented in Fig. 2 (up-right). This family from one side ends (frame 8) with homoclinic asymptotic orbit on L_5 and from the other tends to collision with the primaries (frame 1). The asymmetric periodic orbit of frame 2 (up-right) is a critical periodic orbit since a new branching family of asymmetric periodic orbits bifurcates from it (for critical orbits see Hénon, 1965). Some typical members of this new family, say *asym3b*, are presented in Fig. 2 (down). This family from one side tends to collision with the left primary body (frame 3-down) and from the other tends to collision with the right primary (the reflection of the orbit of frame 3 in the y -axis). From Figs. 1, and 2 we see that the members of these three families are not all simple-symmetric (one intersection with y -axis at half period) periodic orbits. There are members of these families which are higher-multiplicity symmetric periodic orbits. This happens because as the characteristic curves of these families tend, asymptotically, to the triangular equilibrium point L_5 spirally, intersect the y -axis infinity times. In contrary, the bifurcating family *asym3b* (Fig. 2-down) consists of periodic solutions which have always two intersections with y -axis. We studied the stability of the asymmetric periodic orbits of the families *asym1*, *asym2* and *asym3* using the isoenergetic horizontal stability parameters a_h , b_h , c_h and d_h (for details see Hénon, 1965) and found that these three families consist, mainly, of unstable asymmetric periodic solutions, but there exist very small, with respect to x , intervals where the families have stable periodic orbits. The bifurcated family *asym3b* has, similarly, unstable periodic solutions and only one small interval with stable asymmetric periodic orbits exists. We found that the orbits close to all these stable asymmetric periodic orbits of the above four asymmetric families are chaotic.

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