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CHAOS CHARACTERIZATION IN HAMILTONIAN SYSTEMS USING RESONANCE ANALYSIS

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Abstract. Frequency analysis has proven to be a very powerful technique to characterize the transition to chaos in dissipative and conservative dynamical systems. Accurate frequency determination however requires very delicate numerical methods that are also computationaly expensive. I present here the first results obtained using resonance analysis, an alternative method for frequency analysis, based an the construction of a difference equation of suitable order, best representing (in the least-squares sense) an observed time series. The signal "resonances" are then obtained as the eigenfrequencies of this difference equation, by solving its characteristic polynomial. The main advantage of the method is that the resonances are not constrained to be purely harmonic, and can model chaotic systems with broad frequency bands. This kind of representation is thus very well suited to chaotic or dissipative time series, and examples from simple Hamiltonian dynamical systems like the standard map will be presented.

1. INTRODUCTION

Frequency analysis, as introduced in [Laskar 1990], has proved to be a very powerful method to characterize dynamical systems and distinguish between quasi-periodic and chaotic behaviors. Indeed, the so-called power spectrum, that is the distribution of energy over frequencies, is completely different between the two cases, consisting of a small number of discrete isolated peaks with well-defined frequencies for quasi-periodic behavior, and broadband continuous spectra for the case of chaotic motion. In the case of Hamiltonian systems with different behaviors as a function of the actions, the evolution of the frequencies with the actions can also be used to detect chaotic zones, for which the frequencies are no more monotone functions of the actions [Laskar *et al.* 1992].

Accurate fundamental frequencies determination is however a very difficult problem if we only have short time spans of the system history. If we know only N discrete successive positions of our system in phase space, standard discrete Fourier analysis methods will give a frequency accuracy of about $1/N$. For the case of sytems with only one degree of freedom, a clever algorithm due to M. Hénon to determine the rotation number can reach a frequency accuracy of $1/N^2$, thanks to the convergence properties of the continued fraction representation of the rotation number [Lega & Froeschlé 1996]. To reach nearly perfect accuracy, a different method has been developed by [Laskar 1990, Laskar *et al.* 1992, Laskar 1994], using a search for the maximum of a *continuous* Fourier transform, followed by successive eliminations of the found frequencies, and re-orthogonalization of the signal components. The method is very accurate and can determine frequencies even with short signal lengths, but is numerically rather heavy and needs successive iterations to determine the different frequencies.

I will now present results obtained using a completely different approach to the problem of the determination of the frequencies, using a form of the very old Prony method [Prony 1795] to determine exponential components in a signal, and applied here to find *complex* resonances in a signal, that can have both a frequency and a frequency width. The method will be applied to the classical example of the standard map, and will be found to distinguish very easily between quasi-periodic and chaotic zones, even with only a small number of system time history samples.

2. PRONY'S METHOD

A complete description of the application of Prony's method for resonance and frequency determination to the case of Hamiltonian system will be found in [Noullez 2008], and the essence of the technique can be found e.g. in [Hamming 1973]. The problem that we seek to solve is to find the p frequencies ν_k and amplitudes α_k from the (possibly complex) signal

$$u[n] = u(n\Delta t) = \sum_{k=1}^{p} \alpha_k \, \mathrm{e}^{\mathrm{i}2\pi\nu_k \, n\Delta t} \tag{1}$$

sampled at equidistant points in time Δt (that we will now take as unit of time in the formulas). Prony was in fact interested in the expansion of gases, that he was trying to represent as a sum of a finite number of damped *real* exponentials, but the crux of the method is valid for oscillating *complex* exponentials. The important realization of Prony is that any signal that is a simple sum of exponentials has to obey a *linear* constant coefficients difference equation

$$u[n] + a_1 \, u[n-1] + \ldots + a_p \, u[n-p] = 0 \ , \tag{2}$$

and the (complex) "resonances" $\rho_k \equiv \mathrm{e}^{\mathrm{i}2\pi\nu_k}$ are the roots of the characteristic polynomial

$$\rho^p + a_1 \, \rho^{p-1} + \ldots + a_{p-1} \, \rho + a_p = 0 \tag{3}$$

of this difference equation [Hamming 1973].

This means that we can find the so-called *prediction coefficients* a_j by finding the solution of the *linear* least-squares problem for minimizing

$$\varepsilon = \sum_{n=p}^{N-1} \left| \sum_{j=0}^{p} a_j \, u[n-j] \right|^2 \ , \tag{4}$$

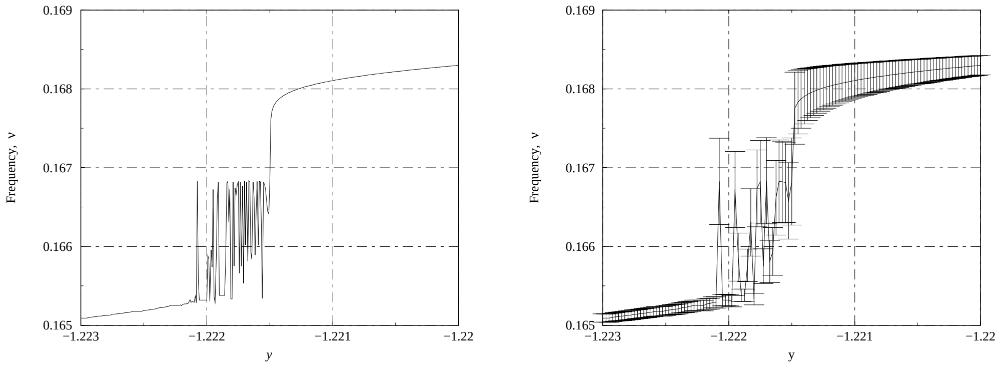

Figure 1: Evolution of the main frequency ν_1 with the action y for the standard map with parameter $a = -1.3$ and initial angle $x_0 = 0$. The plot on the right also shows the corresponding width $\Delta\nu_1$ of the frequency band determined by Prony's method.

and then find the resonances ρ_k as the roots of the polynomial (3) using any standard numerical technique [Hamming 1973], the frequencies being then obtained as $\nu_k = \Im\{\log(\rho_k)\}/2\pi = \arg(\rho_k)/2\pi$. If we are interested in the amplitudes α_k, they can be found by solving a linear system [Hamming 1973] once we know the resonances ρ_k.

The main interest of Prony's method is that it is completely explicit to get simultaneously all prediction coefficients a_j and frequencies ν_k, and it is *exact* if the data really consists of a finite number of exponentials, so that the method is only limited by the computer arithmetic numerical accuracy.

3. TESTS ON THE STANDARD MAP

We have checked Prony's method by reproducing most of the results obtained in [Laskar *et al.* 1992] for the classical standard map with perturbation parameter $a = -1.3$. In particular, we have found that with only 20 successive iterations, we can determine the main frequency with an accuracy better than 10^{-8}, and to full machine accuracy (10^{-16}) with 10^4 points. Prony's method is thus very well suited to determine frequencies very accurately if only a few observations are available.

But the method also gives new informations previously unavailable using the standard frequency analysis method. Indeed, Fig. 1(left) shows the fundamental frequency ν_1 determined by Prony's method for an initial angle $x_0 = 0$ when the action y increases from -1.223 to -1.220 and thus goes through the chaotic zone surrounding the frequency 1/6 hyperbolic point corresponding to $y = -1.221496$. One can see that the frequency evolution is not monotone, implying the presence of a chaotic zone. But at the same time, the resonance ρ_1 found by the method stops to be of unit modulus $|\rho_1| = 1$, as it should be if the corresponding frequency ν_1 was purely real. This indicates that the successive iterates for these values of the action are no more strictly periodic, but present slow variations of the frequency in the interval $\nu_1 \pm \Delta\nu_1$, where $\Delta\nu_1 = -\log(|\rho_1|)/2\pi$ is the width of the band around ν_1 in the frequency spectrum (see Fig. 1(right)). It should be emphasized that this frequency dispersion $\Delta\nu_1$ is determined using a *single* analysis of a signal of sufficient length (typically $N > 1/\Delta\nu_1$), and simply comes from the fact that Prony's method

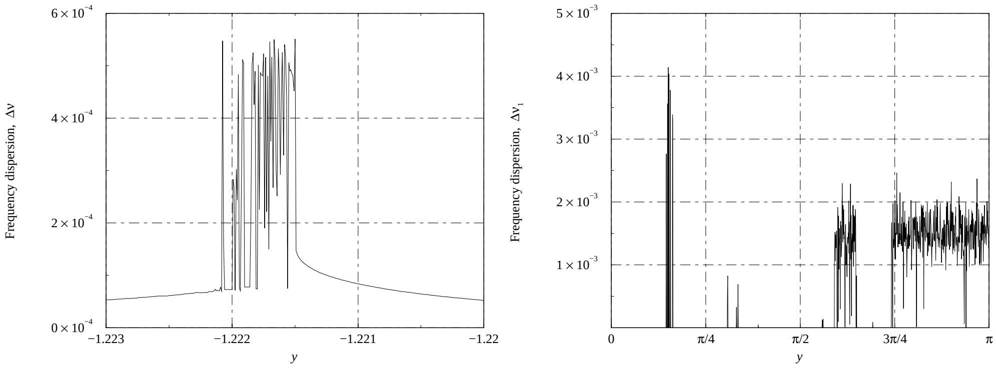

Figure 2: Evolution of the frequency dispersion $\Delta\nu_1$ with the action y. On the right, the same figure for y covering the whole range $[0, \pi]$ and $x_0 = 0$, showing that chaotic regions can be clearly identified.

attempts to model the observations simply as a set of exponentials, that can have both an imaginary part corresponding to pure harmonic oscillations, and a real part corresponding to a damping or a frequency dispersion. In this sense, Prony's method is richer than a simple harmonic analysis as it can represent spectra having both simple peaks corresponding to harmonic frequencies, or broad band features indicating chaotic zones.

The frequency dispersions $\Delta\nu_k$ can thus be used as quite sensitive chaos indicators, showing immediately the non-stationarity of the frequencies ν_k and indicating clearly the presence of chaotic zones, as shown in Fig. 2(left). A complete map of all chaotic regions can be very efficiently obtained, as presented on Fig. 2(right) showing the frequency dispersion $\Delta\nu_1$ for all values of the action y in $[0, \pi]$ and $x_0 = 0$ in the standard map, using a single signal span of length $N = 10^4$ for every value of the action. Prony's method can thus be seen both as a frequency determination technique, but also as a possible tool for chaos determination, complementary with other techniques like Fast Lyapunov Indicators [Froeschlé 1984, Froeschlé *et al.* 1997, 2002]. It could also be a very powerful method for the characterization of *dissipative* systems, where frequencies can be expected to be slightly damped and thus have a corresponding frequency width, that could be perfectly represented in Prony's method.

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