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SINGULARITY ANALYSIS: THE ELEMENTS OF IMPLEMENTATION AND THE ESSENCE OF INTERPRETATION OF THE RESULTS THROUGH A CASE STUDY

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Abstract. Singularity analysis of differential equations is an important tool in the determination of the possible integrability of the equations. Unfortunately the interpretation of the results obtained by the analysis has been and is frequently imperfect. We present the procedure of the analysis and demonstrate the interpretation of the results by means of a system of first-order equations which arise in the modeling of phenomena in divers disciplines.

1. THE LOTKA-VOLTERRA SYSTEM

The dynamical system,

$$\begin{aligned}\dot{x} &= ax + Cxy + zx \\ \dot{y} &= by + Ayz + xy \\ \dot{z} &= cz + Bzx + yz,\end{aligned} \tag{1}$$

was proposed by Lotka (Lotka 25 a) and Volterra (Volterra 26 a, Volterra 31 a). We use the singularity analysis of this system to illustrate several points. To determine the leading-order behavior we substitute $x = \alpha\tau^p$, $y = \beta\tau^q$ and $z = \gamma\tau^r$, where $\tau = t - t_0$ and t_0 is the location of the putative singularity, into (1) and obtain

$$\begin{aligned}\alpha p\tau^{p-1} &= a\alpha\tau^p + C\alpha\beta\tau^{p+q} + \gamma\alpha\tau^{r+p} \\ \beta q\tau^{q-1} &= b\beta\tau^q + A\beta\gamma\tau^{q+r} + \alpha\beta\tau^{p+q} \\ \gamma r\tau^{r-1} &= c\gamma\tau^r + B\gamma\alpha\tau^{r+p} + \beta\gamma\tau^{q+r}.\end{aligned} \tag{2}$$

At least two of the terms in each equation must balance. There are three cases.

1. $p = q = r = -1$
2. $p = q = r = -n$, where $n \geq 2$

3. $p = q = -1$ and $r \geq 0$ *et cyc.*

We firstly consider Cases 2 and 3 since the bulk of the analysis applies to Case 1.

1. 1. CASE TWO

We substitute $x = \sum_{i=0} \alpha_i \tau^{-n+i}$, $y = \sum_{i=0} \beta_i \tau^{-n+i}$ and $z = \sum_{i=0} \gamma_i \tau^{-n+i}$. Balance occurs in the third and fourth terms of each of the equations in (2). The coefficients of the lowest powers in each equation give

$$\alpha(C\beta + \gamma) = 0, \quad \beta(A\gamma + \alpha) = 0 \quad \text{and} \quad \gamma(B\alpha + \beta) = 0, \tag{3}$$

where $\alpha_0 = \alpha$, $\beta_0 = \beta$ and $\gamma_0 = \gamma$. There are two classes of possible cases.

(i): If $\alpha = 0$ from the first equation, this implies that $\beta\gamma = 0$. Hence $\beta = 0$ and $\gamma \neq 0$ or *vice versa*[1]. At the next power, under the assumption that $\alpha = \beta = 0$ and $\gamma \neq 0$, it is necessary to set $\gamma = 0$ and so this case is not a case.

(ii): We assume that $\alpha \neq 0$. From the first of (3) we have $\gamma = -C\beta$. The second gives either that $\beta = 0$ and hence $\gamma = 0$, which returns us to the previous case, or that $\alpha = AC\beta$. In turn this gives either the trivial case or imposes the requirement that $ABC = -1$, *ie* the price of an arbitrary value of β is a restriction on the generality of the system (1). In the following we take $n = 2$ to be specific, but there is no essential difference in the argument. The coefficients for the next power (now -3) are $C\beta_1 + \gamma_1 = -2$, $\alpha_1 + A\gamma_1 = -2$ and $-\dfrac{\alpha_1}{AC} + \beta_1 = -2$ which system may be solved for, say, β_1 and γ_1 in terms of α_1. The price to pay is a further constraint in that now $C = 1 - 1/A$. This with the previous constraint means that $A \neq 1$. In like manner we can solve for β_2 and γ_2 in terms of α_2 from the first two equations of the next triplet. When the third equation is simplified, we find the constraint $[A(a-c) - b + c]\beta = 0$. So far we maintain three arbitrary constants, β, α_1 and α_2 provided $A(a-c) - b + c = 0$ which can be solved for, say, a since A cannot be zero. The next term in the expansion requires $b = c$ and so $a = c$. The system becomes increasingly restricted.

1. 2. CASE THREE

From (2) the coefficients of the leading-order terms give

$$\alpha(C\beta + 1) = 0 \quad \beta(\alpha + 1) = 0 \quad \text{and} \quad \gamma(B\alpha + \beta) = \gamma r \tag{4}$$

and again we discern two cases.

(i): $\alpha = 0$, $\beta = 0$ and γ is arbitrary which implies that $r = 0$. Otherwise $\gamma = 0$ and r is arbitrary.
The two possibilities can be treated togehter. To determine the resonances we write

$$\begin{array}{lcl} x = 0 + \mu\tau^{-1+s} & & \mu(s-1)\tau^{-2+s} = 0 \\ y = 0 + \nu\tau^{-1+s} & \Longrightarrow & \nu(s-1)\tau^{-2+s} = 0 \\ z = \gamma\tau^r + \xi\tau^{r+s} & & 0 = (B\mu + \nu)\gamma\tau^{1+r+s} \end{array}$$

[1]Obviously the cyclic possibilities also occur, but there is no difference apart from the notation. This applies throughout our treatment and we make no further mention of it.

in which we keep only terms linear in the parameters μ, ν and ξ. A nontrivial solution is obtained if we set $s = 1$. Then $\nu = -B\mu$ and ξ is arbitrary[2].

To test for the consistency of the full system (1) we substitute

$$x = \sum_{i=0} \alpha_i \tau^i, \quad y = \sum_{i=0} \beta_i \tau^i \quad \text{and} \quad z = \sum_{i=r} \gamma_i \tau^{r+i}. \tag{5}$$

The only possible value for the free exponent is $r = 0$ so that the two possibilities conflate into one. The coefficients, α_i, β_i and γ_i, $i = 1, \ldots,$ are expressed in terms of α, β and γ. It is interesting to note that there is no singularity and the Taylor expansion of the solution has been obtained by the methods of singularity analysis[3].

(ii): $\alpha = -1$ and $\beta = -1/C$.
The third equation reduces to $\gamma(-B-1/C) = 0$. Thus either $\gamma = 0$ or $B+1/C = 0$. To determine the resonances we write

$$\begin{array}{lcl} x = -\tau^{-1} + \mu\tau^{-1+s} & & \mu(s-1) = C(-\nu - \mu/C) \\ y = -\tau^{-1}/C + \nu\tau^{-1+s} & \Longrightarrow & \nu(s-1) = -\nu - \mu/C \\ z = \gamma\tau^r + \xi\tau^{r+s} & & 0 = B(-\xi + \gamma\mu) + \gamma\nu - \xi/C \end{array}$$

in which factors of τ^{-2+s}, τ^{-2+s} and τ^{-1+r+s} have been removed, respectively.

In the case that $\gamma = 0$ the resonances are at $s = \pm 1$ and the coefficients are $\mu = -C\nu$ and $\xi = 0$. We note that we are short of a constant of integration. To check for consistency we substitute (5) into (1) with now the summation for x and y commencing at -1. The system maintains consistency with one constant short provided of $a = b$. The coefficients of the expansion for z remain as zero, *ie* the system is reduced from a three-dimensional system to a two-dimensional system.

When instead we impose the constraint $B+1/C = 0$, we again have $\alpha = -a-C\beta$ as for the case $\gamma = 0$. Consistency requires that $a = b$ and $r = 0$ to avoid a reversion to the previous case. There are no further requirements for consistency. We obtain a full solution to a reduced problem.

1. 3. **CASE ONE**

The first, third and fourth terms in each equation of (1) are dominant. There are four possible sets of coefficients for the leading-order terms. They are $(\alpha, \beta, \gamma) =$ (i) $(0, -1, -1/A)$, (ii) $(-1/B, 0, -1)$, (iii) $(-1, -1/C, 0)$ and (iv)

$$\left(\frac{A - CA - 1}{1 + ABC}, \frac{B - AB - 1}{1 + ABC}, \frac{C - BC - 1}{1 + ABC}\right)$$

[2]The ‘generic’ resonance of $s = -1$ does not occur as there is no term containing a negative exponent.

[3]The apparent four arbitrary constants – the arbitrary value of t_0 is also to be included – are just that since there are no singular terms in the expansion.

provided $A \neq 0$, $B \neq 0$, $C \neq 0$ and $ABC + 1 \neq 0$, respectively. To determine the resonances we substitute $x = \alpha\tau^p + \mu\tau^{p+s}$, $y = \beta\tau^q + \nu\tau^{q+s}$ and $z = \gamma\tau^r + \xi\tau^{r+s}$ into the dominant terms of (1). The coefficients, μ, ν and ξ, are solutions of the system

$$\begin{bmatrix} s-1-C\beta-\gamma & -C\alpha & -\alpha \\ -\beta & s-1-A\gamma-\alpha & -A\beta \\ -B\gamma & -\gamma & s-1-B\alpha-\beta \end{bmatrix} \begin{bmatrix} \mu \\ \nu \\ \xi \end{bmatrix} = \mathbf{0} \tag{6}$$

the characteristic equation of which is $(s+1)\left(s^2 - s - \alpha\beta\gamma(ABC+1)\right) = 0$ in the case of the fourth set of coefficients. The generic root, $s = -1$, is evident. For general values of the parameters, A, B and C, the second and third roots are not consistent with the desired property of having integral resonances. However, we can look at the special values given for the coefficients of the leading- order terms above.

We consider the case $(0, -1, -1/A)$ and ask the question whether there can exist a solution in terms of a Laurent Series with the requisite number of arbitrary constants after the analysis has shown that the coefficient of the leading-order term of one of the variables is zero. Indeed a very specific question is whether we expect x to produce only zero coefficients *ad infinitum* or can it in fact give rise to a nonzero solution? The second and third resonances are $s = 1$ and $s = 1 - C - 1/A$. Whenever the third resonance is an integer, the analysis can proceed. Provided there is consistency in the complete system a positive integer leads to a Right Painlevé Series valid over a punctured disc centred on the singularity. In the case of a negative integer (less than one) one obtains a Laurent Series valid over an annulus since there are both positive and negative resonances (Andriopoulos 06 a).

As we wish to illustrate some points and the arbitrariness of the values of the parameters makes it difficult to continue the analysis, we take some specific values. We set $A = 1$ and $C = -2$. Then the resonances are $s = -1, 1, 2$ so that we are in the regime of a Right Painlevé Series. We substitute the series

$$x = \sum_{i=0} \alpha_i \tau^{-1+i}, \quad y = \sum_{i=0} \beta_i \tau^{-1+i} \quad \text{and} \quad z = \sum_{i=0} \gamma_i \tau^{-1+i}$$

into the full system, (1), with A and C as above. For the terms relevant to the resonances we obtain

$$\begin{aligned} x &= \alpha_0\tau^{-1} + \alpha_1 + \alpha_2\tau + \ldots \\ y &= \beta_0\tau^{-1} + \beta_1 + \beta_2\tau + \ldots \\ z &= \gamma_0\tau^{-1} + \gamma_1 + \gamma_2\tau + \ldots, \end{aligned}$$

where the constants in the expansions are given by

$$\alpha_0 = -\frac{2}{2B-1}, \quad \beta_0 = \frac{1}{2B-1} \quad \text{and} \quad \gamma_0 = \frac{3-2B}{2B-1},$$

which is fine as $s = 0$ is not a resonance,

$$\alpha_1 = -\frac{-2a + 5b - 2aB - 6bB + 4aB^2 - 9c + 12Bc - 4B^2c}{2(-3+2B)(-1+2B)}$$

$$\beta_1 = -\frac{-2a - b + 10aB - 2bB - 4aB^2 + 9c - 12Bc + 4B^2c}{4(-3+2B)(-1+2B)},$$
$$\gamma_1 = -\frac{-2a - b + 2aB + 6bB - 4aB^2 - 3c - 4Bc + 4B^2c}{4(-1+2B)},$$

in which we note that there is no arbitrary constant, and

$$\alpha_2 = \left(b^2\left(-5 + B\ldots\right)\right) / \left(24(3-2B)^2(-1+B)(-1+2B)(1+2B)\right),$$
$$\beta_2 = \left(4a^2(-1 + B\ldots\right) / \left(48(3-2B)^2(-1+B)(-1+2B)(1+2B)\right),$$
$$\gamma_2 = \left(b^2(13 + B\ldots\right) / (48(-1+B)(-3+2B)(-1+2B)(1+2B)),$$

where the numerators are longs strings of the parameters of the system and again we see no arbitrary constant. A solution consistent with the values of the parameters has been obtained, but this solution contains no arbitrary parameters apart from the location of the simple pole. Consequently the set of initial conditions is severely restricted. We have consistency but at a severe price!

2. CONCLUSION

We have emphasised the delicacy of singularity analysis in the treatment of systems of differential equations when one is concerned with the determination of all possible solutions. The possibility that one or more of the variables may have no singular behaviour is very real. We have seen that in the cases for which a singular behaviour is indicated a variable may have a zero coefficient for the leading-order term. It is also apparent that there are many possibilities and that all of these possibilities deserve investigation when one is interested in the existence of solutions at least in series form and not in the strict sense of a Laurent series. From time to time it bodes us well to examine the sources of practices from two points of view. One is that we have not 'simplified' the theory. The other is that the original theory was actually too simple.

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