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INVERSE PROBLEM, PRECESSING ORBITS AND AUTONOMOUS FORCES

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Abstract. In the framework of the inverse problem of Dynamics we investigate the compatibility between a two-parametric family of precessing orbits and autonomous force fields. In a previous work, for a specific choice of the two parameters in the precessing family of orbits, we obtained that if the family is compatible with a central force, than this is derived from a Manev's type potential. In this work we consider another choice of the parameters in the two-parametric family of precessing orbits, for which we know that there is no central force compatible and study its compatibility with an autonomous force field.

1. INTRODUCTION

According to Kepler's first law, the orbits of planets around the Sun should be ellipses, having the Sun at one of the foci. This law does not take into account the influences of the other bodies, producing a slow rotation of these ellipses in their planes. Even so, there are small differences between the observed precession of planetary orbits and the one computed based on the Newtonian mechanics. For Mercury, the difference is of about $43''$/century.

Using the *Inverse Problem Theory* we search the potential responsible for producing such orbits. In this framework, Xanthopoulos and Bozis (see Xanthopoulos and Bozis 1983) showed that the potential that produces the family of co-planar ellipses with a common focus and arbitrary eccentricity, magnitude and orientation of their major axes has to be the Newtonian potential, in other words, they were able to show that it is possible to deduce Newton's gravitational law using only the first Kepler's law (applied, to several planets, not only to a single one).

The equation of a precessing conic is

$$r(\theta) = \frac{p}{1 + e\cos b(\theta - \theta_0)} \tag{1}$$

where $p = a(1-e^2)$ is the the semilatus rectum, a the semimajor axis, e the eccentricity of the conic section and θ_0 the orientation of the semimajor axis. The rotation of the semimajor axis in the plane of the the conic section is given by b.

We write this equation in the form

$$f(r,\theta,\theta_0) \equiv \frac{p-r}{r\cos b(\theta-\theta_0)} = e \tag{2}$$

and consider it as the equation of a two-parametric family of curves, the two parameters being the orientation of the semimajor axis, θ_0 and the eccentricity e, while the other two parameters (p and b) are assumed to be functions of the first ones.

2. PRECESSING ORBITS AND CENTRAL FORCES

The compatibility between a two-parametric family of orbits and a central force was discussed by Borghero, Bozis and Melis (see Borghero, Bozis and Melis 1999), paper in which they obtained the conditions verified by a two- parametric family of orbits whose members are trajectories for masspoints moving in a central field. These criteria are fulfilled for the family (2) and the force compatible to it is even a conservative one (see Blaga 2005). Its potential turns out to be the potential

$$V(r) = -F_0\left[\frac{b^2}{r} + \frac{p(1-b)^2}{2r^2}\right], \tag{3}$$

known as Manev's potential.

Remark. For $b = 1$ the family of orbits (1) becomes a family of orbits without precession and the potential (3) reduces to the Newtonian one.

If we choose b or p as parameter in (2) we get a negative answer to the question concerning the compatibility of this two parametric family of orbits with a central force. There is *no* central force that admits as orbits all the members of the two parametric family

$$f(r,\theta,b) \equiv r[1+e\cos b(\theta-\theta_0)] = p \tag{4}$$

with b and p parameters, and fixed e and θ_0.

3. TWO-PARAMETRIC FAMILIES OF ORBITS AND AUTONOMOUS FORCES

But are there any autonomous dynamical systems which generates this two- parametric family of orbits? The compatibility between a two-parametric family of orbits and an autonomous conservative dynamical system was discussed by Bozis (Bozis 1983), paper in which the author gave criteria to test the existence of the solution and, in the case of a positive answer, a method to compute it.

We consider the family of planar precessing orbits in Cartesian coordinates

$$f(x,y,b) \equiv (x^2+y^2)[1+e\cos b(\theta-\theta_0)]^2 = p^2 \tag{5}$$

with $\theta = \arctan(y/x)$, b and p parameters, and fixed e and θ_0.

If there is a dynamical system which generates this family of orbits, then the force components $X(x,y)$ and $Y(x,y)$ are related to the family of orbits through:

$$-X_x + \frac{1}{\gamma}X_y - \gamma Y_x + Y_y = \lambda X + \mu Y \tag{6}$$

where the coefficients of the equation are

$$\lambda = \left(-\Gamma_x + \frac{1}{\gamma}\Gamma_y\right)\Gamma^{-1} \qquad \text{and} \qquad \mu = \lambda\gamma + \frac{3\Gamma}{\gamma}, \tag{7}$$

and

$$\gamma = \frac{f_y}{f_x} \qquad \text{and} \qquad \Gamma = \gamma\gamma_x - \gamma_y. \tag{8}$$

If the dynamical system is a conservative one, then $X_y = Y_x$.

The coefficients of the equation (6) are, generally, functions of x, y and b, but we seek solutions $X(x, y)$ and $Y(x, y)$ independent of b, in other words

$$X_b = Y_b = 0\,. \tag{9}$$

Using this condition, Bozis (see Bozis 1983) classified the cases that can arise and gave an algorithm to find the solution if it exists. We need to introduce the functions

$$L = -\frac{\gamma^2}{(1+\gamma^2)\gamma_b}\lambda_b, \qquad \text{and} \qquad M = -\frac{\gamma^2}{(1+\gamma^2)\gamma_b}\mu_b\,. \tag{10}$$

The compatibility of the family of orbits (5) with a conservative autonomous dynamical system is ruled by the following

Proposition. *If*

$$L_b \neq 0\,, \qquad M_b \neq 0\,, \qquad \left(\frac{L}{M}\right)_b \neq 0, \tag{11}$$

and the conditions $\left(\frac{L_b}{M_b}\right)_b = 0$ *and* $\left(\frac{L+M\rho-\rho_x}{\rho}\right)_y = (L+M\rho)_x$, *with* L *and* M *given by (10), and* $\rho = -\frac{L_b}{M_b}$, *are fulfilled, then the problem admits a solution which is found solving the equations:*

$$X_x = \left(-\frac{D}{L_b} - \frac{\rho_x}{\rho}\right)X\,, \quad X_y = \frac{D}{M_b}X\,, \quad Y = \rho X\,, \tag{12}$$

where $D = LM_b - ML_b$. *If one of the above conditions is not satisfied, then no solution exists.*

4. THE TWO-PARAMETRIC FAMILY OF PRECESSING ORBITS

For the family (4) we find

$$\gamma = \tan(\theta - \omega) \tag{13}$$

and

$$\Gamma = -\frac{\cos^2\omega}{\sqrt{x^2+y^2}\,\cos^2(\theta-\omega)} \cdot \frac{1 + e\cos\theta' - b^2 e\cos\theta'}{cos\theta + e\cos\theta\cos\theta' + be\sin\theta\sin\theta'} \tag{14}$$

where

$$\tan\omega = \frac{be\sin\theta'}{1+e\cos\theta'} \quad \text{and} \quad \theta' = b(\theta - \theta_0)\,. \tag{15}$$

To simplify the computation of the functions needed in our analysis, we observe that the function f from (4) is homogenous of degree one in x and y. The function $\gamma = f_y/f_x$ is homogeneous of degree zero in x and y and we can consider it as a function in one variable $z = y/x$. Denoting by $\dot\gamma$ and $\ddot\gamma$ the first and second derivative with respect to z, we obtain that

$$\lambda = \frac{(\gamma z + 1)\ddot\gamma + z\,\dot\gamma^2 + 2\gamma\,\dot\gamma}{x\,\gamma\,\dot\gamma} \quad \text{and} \quad \mu = \frac{(\gamma z + 1)(\ddot\gamma\gamma - 3\dot\gamma^2) + z\,\gamma\,\dot\gamma^2 + 2\gamma^2\,\dot\gamma}{x\,\gamma\,\dot\gamma} \tag{16}$$

5. RESULTS AND CONCLUSIONS

After tedious but straightforward calculations we find that

$$\frac{d\gamma}{dz} = \frac{\cos^2\theta\,\cos\omega}{\cos(\theta-\omega)} \cdot \frac{1 + e\,(1-b^2)\,\cos\theta'}{(\cos\theta + e\,\cos\theta\,\cos\theta' + e\,b\,\sin\theta\,\sin\theta')} \tag{17}$$

and

$$\frac{d^2\gamma}{dz^2} = \frac{e\,b\,\cos^3\theta}{\cos(\theta-\omega)} \cdot \frac{c_3 b^3 + c_2 b^2 + c_1 b + c_0}{(\cos\theta + e\,\cos\theta\,\cos\theta' + e\,b\,\sin\theta\,\sin\theta')^2} \tag{18}$$

where

$$c_3 = e\,\cos\omega\,\sin 2\theta\,, \qquad c_2 = e\sin 2\theta\,\cos\omega + \sin\theta'\,\cos\theta\,\cos(\theta+\omega)\,, \tag{19}$$

$$c_1 = -\sin 2\theta\,\cos\omega\,(\cos\theta' + e)\,, \quad c_0 = -2\,\cos\omega\sin\theta'\,(e\,\cos\theta' + 1)\,. \tag{20}$$

Replacing these functions in (16) we find that the functions λ, μ, L and M depend on b directly and through θ' and ω. It is now straightforward to find that L_b, M_b, $(L/M)_b$ and $(L_b/M_b)_b$, functions which are not zero. According to the proposition of section 3 these results imply that there is *no* autonomous conservative dynamical system compatible with the two-parametric family of orbits (4).

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