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CERTAIN ASPECTS OF THE INVERSE AND THE DIRECT PROBLEM OF DYNAMICS

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Abstract. We present a version of the inverse problem of Dynamics dealing with one material point moving in an inertial frame, in two or three dimensions. We are given monoparametric or two-parametric families of orbits and we require the autonomous force field (conservative or not) which can produce these orbits, for adequate initial conditions. We give the basic tools (PDEs) to face this problem and make certain comments on their applicability. With the above PDEs we then look at the question of integrability of two dimensional potentials from the inverse and the direct problem viewpoint. We also discuss shortly how the three-dimensional inverse problem can help in establishing criteria for the totality of two-parametric spatial families of orbits which can be traced (in the presence of a Newtonian potential) in the exterior of any material distribution.

1. INTRODUCTION

There are several versions of inverse problems in Physics (e.g. Borghero and Bozis, 2005 and 2006) and quite a few even in the framework of Classical Mechanics. The problem which we consider here is the following:We are given a monoparametric set of planar curves

$$f(x, y) = c \tag{1}$$

We want to find all potentials

$$V = V(x, y) \tag{2}$$

which can produce as orbits the members of the given family. It is assumed that, for adequate initial conditions, each of these orbits is traced (partly or totally) in an inertial frame of reference by one material point of unit mass, under the action of this potential.

There appear in the literature more general results for motion (i) in autonomous and non-autonomous force fields (Anisiu, 2004), (ii) in rotating frames (Bozis and Stavinchi, 1996; Bozis and Anisiu, 1996), (iii) in three dimensions (Shorokhov, 1988;

Puel, 1992; Bozis and Kotoulas, 2005; Anisiu, 2005; Kotoulas and Bozis, 2006) and (iv) in holonomic systems with n degrees of freedom (Melis and Borghero, 1986; Bozis and Borghero, 1995; Borghero, 1999). In what follows, instead of giving proofs of formulae, we shall comment on the meaning and on the ground of applicability of these formulae and the pertinent results.

In section 2 we present the basic formulae and the pertinent notation. In section 3, the direct problem is discussed, basically in connection to the question of integrability. In section 4, we remind the reader of the basic tools of the inverse problem in three dimensions and we give notice of the presence of a new free-of-the-energy second order PDE which, combined with Laplace's PDE, can help essentially in establishing criteria for the totality of orbits traced outside any material concentration. Finally, in section 5 we give a short account of some open questions.

2. BASIC FORMULAE

Newton is frequently quoted as the first who faced the inverse problem in dealing with Kepler's laws. There followed Bertrand, Darboux, Dainelli, Joukovsky and others (Whittaker, 1944). Back in the seventies, Szebehely (1974) and even earlier Dramba (1963) have given a formula relating the family (1), the potential (2) and the energy-dependence function $E = E(f(x, y))$ along the members of (1).

a. This formula was modified (Bozis, 1983b in Bozis, 1995) to look like

$$V_x + \gamma V_y + \frac{2\Gamma}{(1+\gamma^2)}(E - V) = 0 \tag{3}$$

where

$$\gamma(x, y) = \frac{f_y}{f_x} = -\frac{\dot{x}}{\dot{y}} \quad \text{and} \quad \Gamma = \gamma\gamma_x - \gamma_y \tag{4}$$

The requirement $E_y = \gamma E_x$ leads to the free-of-the-energy, second order in $V(x, y)$, PDE

$$-V_{xx} + kV_{xy} + V_{yy} = \lambda V_x + \mu V_y \tag{5}$$

where

$$k = \frac{1-\gamma^2}{\gamma} \quad , \quad \lambda = \frac{\Gamma_y - \gamma\Gamma_x}{\gamma\Gamma} \quad , \quad \mu = \lambda\gamma + \frac{3\Gamma}{\gamma} \tag{6}$$

Equation (3) is linear in V. There is an one-to-one correspondence of slope functions $\gamma(x, y)$ and families (1). Pairs (γ, V) of families and potentials which satisfy the PDE (5) are called *compatible.* For any compatible pair (γ, V) the energy E can be found from (3).

b. What if we are given a two (or three)-parametric family?

$$f(x,y,c_2)=c_1, \quad or \quad f(x,y,c_2,c_3)=c_1 \tag{7}$$

In general, no solution exists, unless the given family satisfies certain differential conditions, in which case the solution is unique (Bozis, 1983b and Xanthopoulos and Bozis, 1983 in Bozis, 1995).Thus, e.g. (i) the family of concentric circles $r=c$ can be generated by any of the potentials $V(r,\theta)=g(r)+\frac{1}{r^2}h(\theta)$ (g,h arbitrary functions, r,θ polar coordinates) and is traced with energy $E=g(r)+\frac{1}{2}rg'(r)$, accordingly. (ii)The two-parametric family of conic sections $r=\frac{c_1}{1+c_2\cos\vartheta}$ is generated only by the Newtonian potential $V=-\frac{1}{r}$, whereas (iii) no potential can produce the two-parametric family $r=\frac{c_1}{1+c_2\cos^2\vartheta}$.

c. Is the entire given family (1) actually traced by the material point in the presence of any potential? No! Only some members or parts of certain members are traced *as real orbits.* These are lying in the domain of the xy plane defined by the so-called *family boundary curves* (FBC)

$$\frac{V_x+\gamma V_y}{\Gamma}\leq 0 \tag{8}$$

(Bozis and Ichtiaroglou, 1994 in Bozis, 1995). It can be checked e.g. that the potential $V(r,\theta)=\ln r+\frac{1}{2r^2}\cos 2\theta$ produces full circles or arcs of circles, traced with energy $E=\ln r+\frac{1}{2}$ outside the Bernoulli's lemniscate $r^2\geqslant\cos 2\theta$. As they refer to orbits of various energies, the FBC do not generally coincide with the known ZVC (e.g. Szebehely, 1967).

d. As seen from (3), for $\Gamma=0$, it is $\gamma=-\frac{V_x}{V_y}$. Then, from (4), we get

$$V_{xy}(V_x^2-V_y^2)=V_xV_y(V_{xx}-V_{yy}) \tag{9}$$

giving all potentials which can produce families of straight lines (Bozis and Anisiu, 2001).

e. As the PDE (5) is not generally solvable, homogeneity plays a decisive role in solving it. In fact, if $V(x,y)=x^mR(z)$ and $f(x,y)=x^rT(z)$, $z=\frac{y}{x}$ then $\gamma=\gamma_0(z)$ and the PDE (5) becomes *ordinary* linear DE in $R(z)$ of the second order. Homogeneity of the potential combined with isoenergeticity of the compatible families simplifies even more the problem (Borghero and Bozis, 2002).

f. A generalization of formula (5) refers to autonomous force fields $\mathbf{X}=\mathbf{X}(\mathbf{x},\mathbf{y})$, $\mathbf{Y}=\mathbf{Y}(\mathbf{x},\mathbf{y})$. It reads

$$Y_y-X_x+\frac{1}{\gamma}X_y-\gamma Y_x=\lambda X+\mu Y \tag{10}$$

An interesting application of this formula answers the following question: How second order linear ODEs (solvable or not)

$$y'' + a(x)y' + b(x)y = h(x) \tag{11}$$

are related to force fields? Bozis and Borghero (2008) have shown that all the solutions of (11) (constituting a two-parametric set) can become orbits of (at least) one autonomous force field. It was found that, e.g., for $e_0 = const., a \neq 0, h \neq 0$, this field is given by

$$X = e_0 a J^2, \quad Y = e_0(h - yb)J^2 \quad \text{where} \quad J = \exp(\int a dx) \tag{12}$$

So, force fields may be thaught of as mechanical devices which solve numerically (as e.g. the Runge-Kutta procedure does) linear ODEs of the second order which are not analytically solvable.

3. DIRECT PROBLEM. INTEGRABILITY

a.The basic PDE (5) comes also in the form

$$\begin{aligned} & (V_x + \gamma V_y)(\gamma^2\gamma_{xx} - 2\gamma\gamma_{xy} + \gamma_{yy}) \\ = \quad & \Gamma\left\{-\gamma_x V_x + (2\gamma\gamma_x - 3\gamma_y)V_y + \gamma(V_{xx} - V_{yy}) + (\gamma^2 - 1)V_{xy}\right\} \end{aligned} \tag{13}$$

which is more suitable for direct problem considerations.

To the question: "For a given *V(x,y)*, how many families $\gamma(x, y)$ do we get?", a physicist may be surprised to hear a mathematician's answer: As many monoparametric families as two arbitrary functions allow. The physicist knows that the totality of orbits constitutes a three-parametric family $F(x, y, c_1, c_2, c_3) = 0$ and, in order to explain the apparent enigma, he has to think that this totality of orbits may serve to give back the mathematician's set of *monoparametric* families by taking $c_2 = c_2(c_1)$, $c_3 = c_3(c_1)$.

b. If we can find a solution of (13) of the form $\gamma = \gamma(x, y, c_1, c_2)$, then it is as if :
(i) We have shown that the pertinent potential *is integrable* and
(ii) We have found the *second integral*.

Indeed. With the aid of (3), we replace one of the two constants (c_1, c_2) by E, then we put the other constant equal to Φ and so we come to obtain the so-called *orbital* function $\gamma = \gamma(x, y, E, \Phi)$ (Bozis, 2005) and then, in turn,

$$\Phi = \Phi(x, y, \gamma, E) = \text{constant}. \tag{14}$$

Now, since $\gamma = -\frac{\dot{x}}{\dot{y}}$ and $E = V + \frac{1}{2}(\dot{x}^2 + \dot{y}^2)$, we get

$$\Phi(x, y, \gamma, E) = I(x, y, \dot{x}, \dot{y}) = cons\tan t \tag{15}$$

Since $\dot{x} = -\gamma\dot{y}, \dot{\gamma} = -\Gamma\dot{y}$ and with the aid of (3), we obtain from $\frac{d\Phi}{dt} = 0$, the following substitute for the vanishing of the Poisson bracket

$$2(E-V)(\gamma\Phi_x - \Phi_y) = (1+\gamma^2)(V_x + \gamma V_y)\Phi_\gamma \tag{16}$$

Equation (16) is satisfied by all functions (14).

c. The inverse problem equipment has an outlook on the following question: Why is it and integrable potentials are so rare? After all , Poisson's PDE is linear and homogeneous! How come and it can be solved for $\Phi(x, y, \dot{x}, \dot{y})$ for an *adequate* (integrable) $V(x, y)$ and the same PDE cannot be solved for a slightly changed $V(x, y)$? The inverse problem faces the question from the opposite side, by putting another question:

We are given a function $\varphi = \varphi(x, y, a, b)$, where $a = \dot{x}$, $b = \dot{y}$ and we ask: Is there a potential $V = V(x, y)$ for which the above function *actually is an integral of the motion?*

Let us define

$$\Delta = (a\varphi_x + b\varphi_y) \quad \text{and} \quad P = \frac{\varphi_a}{\Delta}, \quad Q = \frac{\varphi_b}{\Delta} \tag{17}$$

The criteria for an affirmative answer are (Bozis and Ichtiaroglou, 1987 in Bozis, 1995):

$$\begin{aligned} &\frac{P_a}{Q_a} = \frac{P_b}{Q_b} = m, \quad m_a = m_b = 0 \\ &m(P - mQ)_x + (P - mQ)_y = m_x(P - mQ) \end{aligned} \tag{18}$$

If the conditions (18) are satisfied, the potential is determined from

$$V_x = \frac{1}{(P - mQ)}, \quad V_y = -\frac{m}{(P - mQ)} \tag{19}$$

Comment: Two exceptional cases are: (i) $\Delta = 0$ corresponding to the angular momentum and the momentum integral, i.e. to central and one-dimensional potentials, and (ii) $P_a = Q_a = P_b = Q_b = 0$. These cases are also fully studied in the above paper.

In general, to one good $\varphi(x, y, a, b)$ (satisfying the conditions (18)) there corresponds a unique potential (apart from an additive constant). So, if all good $\varphi's$ are collected into one set $S(\varphi)$, to this set $S(\varphi)$ there will correspond another definite set $S(V)$ which includes all the images of the set $S(\varphi)$. These are the integrable potentials. No wonder then if a potential given at random does not happen to be a member of the set $S(V)$! This argument justifies the scarcity of integrable potentials (Bozis, 2007).

As an example we can check that the expression $\varphi = 9b^4 + 12y^2b(3xb - ya) - 2y^4(6x^2+y^2)$ satisfies the above criteria (18) and that to this integral there corresponds the potential $V = \frac{16}{3}x^3 + xy^2$.

d. Darboux's criterion of integrability states that: If, for constant a, b, c, d, e, f and for $A = 2axy+bx+dy+c$, $B = 2\left\{a(x^2-y^2)+dx-by+f-c\right\}$, $C = 3(2ay+b)$, $D = 3(2ax+d)$ and for a given potential $V(x,y)$, there exist six-tuples {a,b,c,d,e,f}$\neq$ $\{0,0,c,0,0,c\}$ such that

$$A(V_{xx}-V_{yy})-BV_{xy}+CV_x-DV_y=0 \tag{20}$$

then the potential $V(x,y)$ is integrable, having an algebraic second integral, of the second degree in the velocity components. Ichtiaroglou and Meletlidou (1990) put the question : Are there families $\gamma(x,y)$ for which equation (20) coincides with (5)? They found that, for *four classes* of families, the answer is affirmative. Thus, e.g., the simplest class is the set of the circles $x^2+y^2=c$, for which $\gamma=\frac{y}{x}$ and $V=g(r)+\frac{1}{r^2}h(\theta)$. The pertinent second integral is $\Phi=(x\dot{y}-\dot{x}y)^2+2h(\theta)$.

e. Not only integrability but also nonintegrability has been considered in the light of the inverse problem. In view of Yoshida's (1987) criterion, Bozis and Meletlidou (1998) have shown that an observed family of geometrically similar orbits may assert nonintegrability *of all or of some* homogeneous potentials (of integer degree m, different from 0 and from ± 2) which can generate the family.

4. THREE-DIMENSIONAL INVERSE PROBLEM

For the two-parametric family of spatial curves

$$f(x,y,z)=c_1, \quad g(x,y,z)=c_2 \tag{21}$$

there exist two Szebehely's type PDEs (relating potentials $V=V(x,y,z)$, families (21) and energy functions $E=E(c_1,c_2)$) (Varadi and Érdi, 1983; Bozis and Nakhla, 1986, in Bozis,1995; Shorokhov, 1988; Puel, 1992). In place of the family (21), we can use *two slope functions* $\alpha(x,y,z)$ and $\beta(x,y,z)$ defined with the aid of the tangent vector $\{\delta_1,\delta_2,\delta_3\}=\nabla f\times\nabla g$ as

$$a=\frac{\delta_2}{\delta_1} \quad , \quad \beta=\frac{\delta_3}{\delta_1} \tag{22}$$

(e.g. Anisiu, 2005).The two PDEs are

$$aV_x-V_y=\frac{2a_0}{\Theta}(E-V) \quad and \quad \beta V_x-V_y=\frac{2\beta_0}{\Theta}(E-V) \tag{23}$$

where $\Theta=1+a^2+\beta^2$, $\quad a_0=\overrightarrow{\epsilon}\nabla a$, $\quad \beta_0=\overrightarrow{\epsilon}\nabla\beta$.

Clearly, as we have *two* equations in *one* unknown, in general, we expect no solution, unless the given family (21) satisfies certain conditions (e.g. Shorokhov, 1988). We obtain also two energy-free PDEs (i) by eliminating the energy between (23) and (ii) by taking into account that $E=E(f(x,y,z),\ g(x,y,z))$. We proceed with

$a_0^2 + \beta_0^2 \neq 0$ (the case $a_0 = \beta_0 = 0$ corresponds to families of straight lines (Bozis and Kotoulas, 2004)) and we obtain

$$\begin{aligned}(a\beta_0 - a_0\beta)V_x - \beta_0 V_y + a_0 V_z &= 0\\ \mu_1 V_{xx} + \mu_2 V_{xy} + \mu_3 V_{xz} + \mu_4 V_{yy} + \mu_5 V_{yz} &= \mu_7 V_x + \mu_8 V_y + \mu_9 V_z\end{aligned} \tag{24}$$

with $\mu_1, \mu_2, ...\mu_9$ depending on the given family. The expected coefficient μ_6 of V_{zz} happens to be zero. The PDEs (24) are good for $a_0 \neq 0$ (For $a_0 = 0$, we apply a transformation and obtain the corresponding to (24) formulae (Bozis and Kotoulas, 2005).The formulae become simpler for potentials which are homogeneous or axially symmetric, combined with orbits which have some simplifying geometrical property (e.g. Kotoulas and Bozis, 2006).

As the first PDE in (24) can offer three PDEs of the second order, the system (24) is equivalent to a system $S_{4,2}$ of *four* PDEs of the second order. It is then Charpit's idea that the compatibility of such a system may imply a fifth PDE which *by necessity* must be true (Smirnov, 1964, Miller, 1956). By requiring compatibility of the 10 third order derivatives of $V(x, y, z)$, we did actually find the PDE

$$s_1 V_{xx} + s_2 V_{xy} + s_3 V_{xz} + s_4 V_{yy} + s_5 V_{yz} + s_6 V_{zz} = s_7 V_x + s_8 V_y + s_9 V_z \tag{25}$$

and we made sure that this fifth equation is *generally independent* of the other four by checking the pertinent Jacobians. Having at our disposal the system $S_{5,2}$, it is natural to complete it by Laplace's PDE $V_{xx} + V_{yy} + V_{zz} = 0$ and obtain a system $S_{6,2}$. Then we can express the six second order derivatives $V_{xx}, V_{xy}, V_{xz}, V_{yy}, V_{yz}, V_{zz}$ in terms of first order derivatives and so we end up with *necessary conditions for the family (21) to be traced outside any material concentration* (Bozis and Koukouloyannis, 2008; in preparation).

5. SOME OPEN QUESTIONS

a. The new equation (25) for the 3-D problem may be combined with the Poisson's PDE $\nabla^2 V(x, y, z) = 4\pi G\rho$ to provide necessary conditions for the family (21) to be traced *inside* any material concentration.

b. The direct problem in 3-D may be considered with *known* the two slope functions (22).

c. Further study of the question of programmed motion on the grounds of the inequality (8) is desirable. An attempt to this direction is made by the contribution of Bozis and Anisiu, in the present Volume.

d. Series expansion methods may be applied to write down the basic equations of the inverse problem for *distorted families of orbits.*

e. Further study is needed for families given in parametric form $x = x(t, b), y = y(t, b)$ with $E = E(b)$ (e.g. Bozis and Borghero, 1998) or in implicit form by $F(x, y, c) = 0$.

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