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THE APPLICATION OF THE FBILI METHOD TO THE SOLUTION OF RADIATIVE TRANSFER PROBLEMS IN MOVING MEDIA

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Abstract. The application of the fast convergent Forth-and-Back Implicit Lambda Iteration (FBILI) method to the solution of the two-level atom line transfer problems in moving media with low velocity fields is presented. Two astrophysically important problems are solved and discussed: (a) line formation in a plan-parallel moving slab of finite thickness, and (b) line formation in a spherically symmetric expanding stellar atmosphere.

1. INTRODUCTION

For the modelling of many astrophysical objects it is necessary to solve the radiative transfer (RT) problem taking into account the motion of the medium. In the media with low velocity regime the radiative transfer is usually solved in the observer's (laboratory) frame of reference. As in the static case, the RT equation 'along the ray' is an ordinary differential equation, but the opacity and emissivity of the material, as seen by the observer at rest, depend on the direction of propagation of radiation due to Doppler effect. Angles and frequencies are coupled together by the Doppler shift. Using the observer's reference frame, most numerical techniques developed for static media can be straightforwardly applied to the RT in moving media with arbitrary (non-monotonic) velocity fields. Only a wider range of frequencies (due to Doppler shifts) and a larger number of angles (due to the coupling between the angle and the frequency) must be used.

For flows with speeds much larger than thermal, radiative transfer is preferably formulated in the Co-Moving Frame (CMF) of reference, although the disadvantage of CMF calculations is the imposition of monotonic velocity fields. In the study of high-speed outflows from stars, supernovae etc., where the velocity gradients greatly enhance the escape of photons, Sobolev (or the large-velocity gradient - LVG) approximation is commonly used (Sobolev 1957).

Here we solve the line formation problem in moving (plane-parallel and spherically symmetric) media by the use of a fast convergent method, Forth-and-Back Implicit Lambda Iteration (FBILI), developed by Atanacković-Vukmanović et al. (1997).

2. LINE TRANSFER IN MOVING MEDIA (IN THE OBSERVER'S FRAME)

We will consider the case of a two-level atom line formation in a spherically symmetric expanding stellar atmosphere (transition to the case of a plane-parallel expanding slab of finite thickness is straightforward). We shall assume that the physical properties vary only with radial distance r. The radiative transfer equation (RTE) in the observer's frame takes the following form:

$$\mu\, \frac{\partial I(r,\nu,\mu)}{\partial r} + \frac{1-\mu^2}{r}\, \frac{\partial I(r,\nu,\mu)}{\partial \mu} = -\,\chi(r,\nu,\mu)\left[I(r,\nu,\mu) - S(r,\nu,\mu)\right]. \qquad (1)$$

$I(r,\nu,\mu)$ is the specific intensity at point r, at frequency ν and in direction μ (cosine of the angle θ between the local outward radial direction and the direction of propagation of radiation at radius r). In moving media, the absorption coefficient $\chi(r,\nu,\mu)$ and the source function $S(r,\nu,\mu)$, as seen by the observer at rest, depend on the direction of propagation of radiation.

Instead of solving the RTE as the partial differential equation (1), we can perform a ray-by-ray computation of the specific intensities along the set of directions tangent to the spherical layers (like those shown in Fig. 1) using the ordinary differential RTE in the 'along the ray' form:

$$\pm\frac{dI^{\pm}(x,\mu)}{d\tau(x,\mu)} = I^{\pm}(x,\mu) - S(x,\mu)\ . \qquad (2)$$

Here, τ represents the optical distance along a given direction (ray) measured from the surface, whereas $I^{\pm}$ are the ingoing and outgoing specific intensities along the ray. The monochromatic optical distance along a given direction is given by

$$d\tau(x,\mu) = -\chi(x,\mu)dz\ \ , \qquad (3)$$

and can be written as

$$d\tau(x,\mu) = d\tau^L\left[\beta + \phi(x,\mu)\right]\ , \qquad (4)$$

where dz is the corresponding geometrical path length, and χ contains both continuum and line contributions. $\beta = \chi^C/\chi^L$, $d\tau^L = -\chi^L(z)dz$, and Φ is the line absorption profile, which is in the case of pure Doppler broadening given as:

$$\phi(x,\mu) = \frac{1}{\delta\sqrt{\pi}}e^{-(x-\mu V)^2/\delta^2}\ , \qquad (5)$$

where $\delta = \Delta\nu_D/\Delta\nu_D^*$ is the ratio of the Doppler widths at a local temperature and at some standard temperature T^*.

In the case of a two-level atom and assuming the complete redistribution, the source function can be writen as:

$$S^L = \epsilon B + (1-\epsilon)\bar{J}\ , \qquad (6)$$

where ε is the photon destruction probability, B is the Planck function and:

$$\bar{J} = \frac{1}{2}\int_{-\infty}^{\infty} dx \int_{-1}^{1} d\mu I(x,\mu)\phi(x,\mu) \qquad (7)$$

is the scattering integral. The total source function is:

$$S(x,\mu) = \frac{\beta}{\beta + \phi(x,\mu)} S^C + \frac{\phi(x,\mu)}{\beta + \phi(x,\mu)} S^L \tag{8}$$

Once the line formation problem is defined by Eqs. (2) and (6)-(8), we can look for its numerical solution.

2. 1. DISCRETIZATION

For the numerical description of the radiation transport through 1D moving spherical atmosphere, a discrete set of radii $\{r_l\}$, $l = 1, n$ is needed (Fig.1). Let the radius r_1 corresponds to the upper boundary surface of the atmosphere. The radius r_n of the lower boundary is to be chosen so that the radiation field at that point is highly isotropic.

The solution of RTE (2) is performed along the set of rays $\{z_k\}, k = 1, n$ tangent to the spherical layers corresponding to the discrete set of radii $\{r_l\}$, as well as along a few additional, so-called core rays $\{z_k\}, k = n+1, nt$ that intersect the inner boundary surface (see Fig. 1).

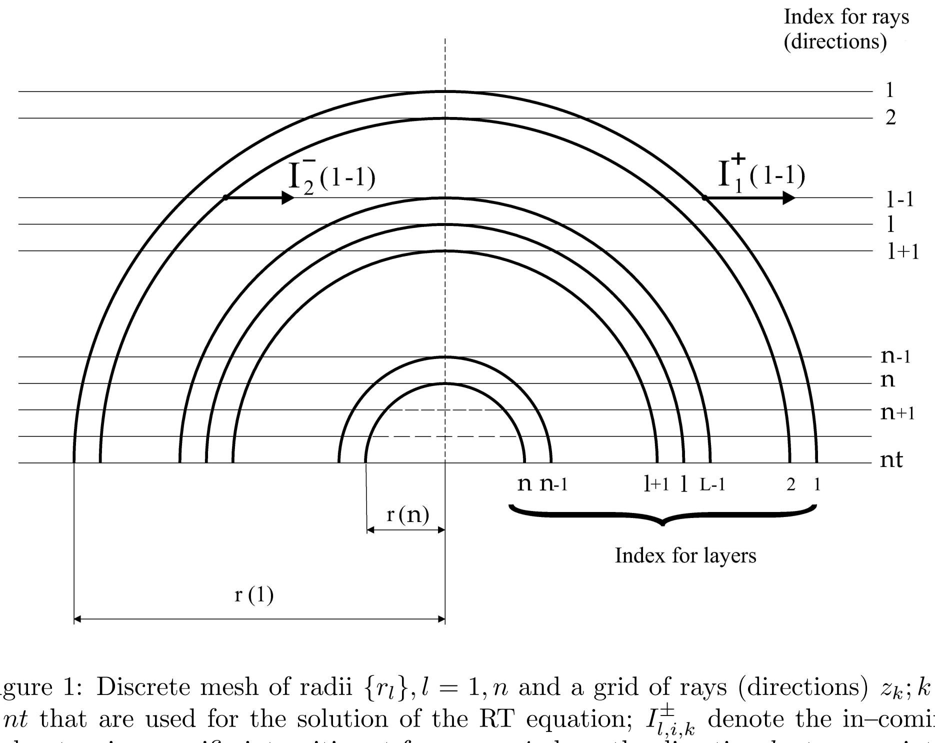

Figure 1: Discrete mesh of radii $\{r_l\}, l = 1, n$ and a grid of rays (directions) $z_k; k = 1, nt$ that are used for the solution of the RT equation; $I^{\pm}_{l,i,k}$ denote the in–coming and out–going specific intensities at frequency i along the direction k at any point l.

3. FBILI SOLUTION OF THE LINE TRANSFER IN MOVING MEDIA (IN THE OBSERVERS FRAME)

Although the radiation field is unknown, using two-stream approximation we can represent its propagation by means of the integral form of the RT for both the in-going and the out-going specific intensities as follows:

$$I_l = I_{l-1} e^{-\Delta} + \int_0^{\Delta} S(t) e^{t-\Delta} dt \ . \tag{9}$$

In order to solve this integral, we will assume parabolic behavior of the source function between two successive depth points. Proceeding in this way we will derive the implicit linear relation between the mean intensity of the radiation field and the local line source function:

$$\bar{J} = a + bS^L \ . \tag{10}$$

This relation is implicit as the value of the source function is also unknown. It depends on the unknown radiation field via scattering processes. By substituting Eq. (10) into SE equation (6), we get the expression for updating the source function:

$$S^L = \frac{\varepsilon B + (1-\varepsilon)a}{1-(1-\varepsilon)b} \ . \tag{11}$$

The iterative computation of these coefficients and not of the unknown functions ($\bar{J}$ and S^L) themselves like in the classical Λ iteration, speeds up the convergence dramatically.

3.1.1. Forward step

We start from the lower boundary condition ($I_1^- = 0$) and then solve the integral (9) for ingoing radiation for $l = 2, NL$. Then we perform numerical integration over directions and frequencies, so we get the mean intensity in the form

$$\bar{J_l}^- = \tilde{a}_l^- + \tilde{b}_l^- S_l + c_l^- S_l' \tag{12}$$

where we put all the non-local quantities in coefficient $\tilde{a}_l^-$.

In order to improve the convergence, we can 'pack' the coefficients differently:

$$b_l^- = \frac{\tilde{a}_l^-}{S_l^L} + \tilde{b}_l^- \ , \tag{13}$$

so we have

$$\bar{J}_l^- = b_l^- S_l^L + c_l^- S_l'^L \ . \tag{14}$$

Now, because $\frac{\tilde{a}_l^-}{S_l^L}$ is the ratio of two homologous quantities, it quickly gets its exact value speeding up the convergence of the whole iterativeprocedure. The coefficients b_l^- and c_l^- are stored for the use in backwards step.

3.1.2. Backward step

Now, we proceed from the bottom layer where the out-going specific intensities are known. For the rays with $k > n$ we use the diffusion approximation or we simply take that $I^+_{n,i,k>n} = S_{n,i,k}$, whereas for $k = n$ the condition $I^+_{n,i,n} = I^-_{n,i,n}$ is to be used. At all other upper points $l = n-1, 1$, we can compute $I^+_{l,i,k}$ using the integral form of the RTE (eq. (15)) for outgoing intensities. Here we also used parabolic approximation for the source function between two depth points. Again, after numerical integration, we will get:

$$\bar{J}^+_l = a^+_l + b^+_l S^L_l \tag{15}$$

Since we have b^-_l and c^-_l, and we know that (with assumption of parabolic behaviour of source function) $S'^L_l = 2\frac{S^L_{l+1}-S^L_l}{\Delta\tau} - S'^L_{l+1}$, we can now get a^-_l and b^-_l. Next, after computing a^+_l and b^+_l in the current layer, we can already update the source function and use this new value for computing the out-going intensities. The whole procedure is repeated until the convergence criterion is satisfied.

4. TEST PROBLEMS AND RESULTS

In order to test the FBILI method when applied to RT in moving media, we solved several benchmark problems.

4. 1. LINE FORMATION IN A PLANE-PARALLEL MOVING SLAB

First we solved the problem of RT in plane-parallel expanding slab of finite thickness (Hummer and Rybicki, 1968). The center of the slab is at rest, whereas the part of the slab closer to the observer is moving towards and the part that is farther is moving away from the observer with the velocity normal to the surface. There is no incident radiation on the boundaries of the slab. This simulates the expanding emission nebula.

The velocity law is given by:

$$V(\tau) = V_0 + \tau V_1,$$

with three values for the velocity gradient: $V_1 = 0$, $V_1 = -0.1$ and $V_1 = -0.2$. The parameters describing the slab are as follows: $\varepsilon = 10^{-3}$, $B = 1$, $T = 20$, $\delta = 1$.

The results we got are in agreement with those in H&R paper. The plot of intensity with frequency is presented on fig 2. The variation of maximum relative change with number of iterations for Jacobi and FBILI method (with and without iteration factor) is presented in Fig. 3.

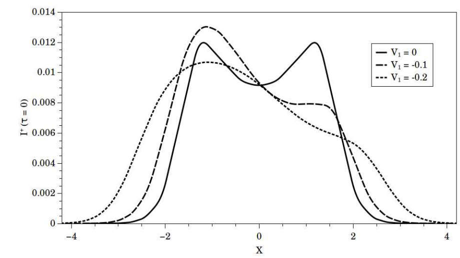

Figure 2: Emergent intensity at $\mu \sim 1$ from an expanding slab ($\varepsilon = 10^{-3}$, $B = 1$, $T = 20$) for three values of the velocity gradient.

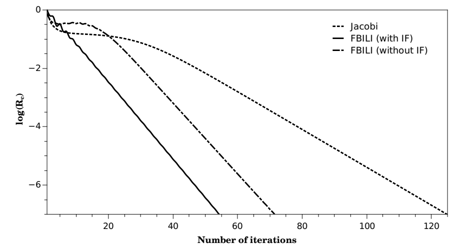

Figure 3: Maximum relative change R_c of the source function between two iterations as a function of iteration number for Jacobi and FBILI method (with and without iteration factor) for the case of line formation in a plane-parallel expanding slab.

4. 2. LINE FORMATION IN A SPHERICAL ATMOSPHERE

The use of the FBILI method for the solution of the line transfer in a spherically symmetric atmosphere is tested on the benchmark problem proposed by Avrett & Loeser (1984).

We consider a stellar atmosphere consisting of homogeneous spherical shells. We take that the radius of the first layer $r_1 = 30$ and the last one $r_n = 1$ (in the units of stellar radius). The opaccity is:

$$\chi(r, x, \mu) = \left[\frac{120}{29} + \frac{30000}{29}\phi(r, x, \mu)\right]\frac{1}{r^2}, \tag{16}$$

where the line profile ϕ is given by the Gaussian profile function (5). The velocity law is given by:

$$V(r) = \frac{6}{\pi}\left[\arctan\left(\frac{2r - 31}{29}\right) + \frac{\pi}{4}\right], \tag{17}$$

so that $V = 0$ at the bottom and $V = 3$ at the surface.

The line profile is presented on fig 4, while comparison of maximum relative change for FBILI and Jacobi method is in Fig 5.

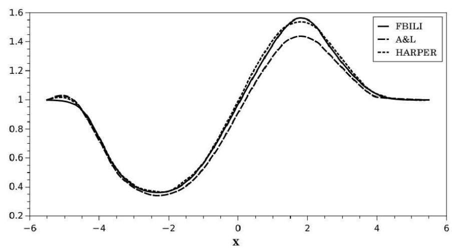

Figure 4: Line profile from an expanding atmosphere with a frequency independent continuum source function.

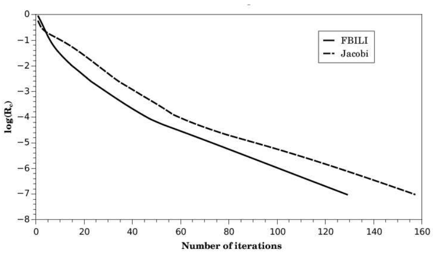

Figure 5: Maximum relative change R_c of the line source function between two iterations as a function of the iteration number for the Jacobi and FBILI method for the case of line formation in an atmosphere expanding according to the velocity law (17).

5. CONCLUSIONS

The obtained results are in a good agreement with the results from four independent investigations performed by other authors. Due to the lack of the published results on the convergence properties of the iterative procedures used by other authors to solve these benchmark problems, we compared the convergence rate of the FBILI to Jacobi method, which uses the same formal solver. When applied to a plane-parallel moving slab, FBILI method is about 2.5-3 times faster. On the other hand, for spherically symmetric expanding atmosphere, the FBILI method is 1.4-1.8 times faster than the Jacobi method, which is a bit less than in the static case, where the convergence is 1.7-2 times faster.

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