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MOVEMENT OF A BODY WITH VARIABLE MASS

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Abstract. In this paper we consider moving bodies with variable mass. Such kind of motion often occurs in nature and technology. We derived an equation of motion of bodies with variable mass in general case and by the example of rocket motion. Basic and some specific equations are given to describe such movement.

1. INTRODUCTION

The earliest recorded work on the dynamics of bodies with varying mass was performed in the eighteenth century by Bernoulli. He was then studying the forces acting on a liquid jet propelled hydroreactive ship - an ancient application of the principle of jet propulsion. He actually derived what may be referred to as the equation of motion in this special case. The Czech scientist and inventor, George von Buquoy[1] was the first to pose the general problem of the dynamics of systems with varying mass. In 1812, he obtained his 'motion formula' for such systems, and went on to solve a large number of examples based on this formula. Von Buquoy's work can be said to mark the birth of the theory of the dynamics of systems with varying mass. In the mean time, William Moore worked out his mathematical theory of rocket motion in England in 1813, and, in 1819, S.D.Poisson[2] took a rather modern approach and derived the equations of motion of variable mass systems based on Lagrange's general formula. In a book published in 1856, Tait and Steele included a section on mass variation. They postulated that mass variation produced small continuous impacts or impulsive forces on systems, and thus resulted in continuous change of velocity. This work was followed, several years later, by that of Meshcherskii, whose work spanned the period 1897 to 1904. He essentially laid the foundation for the development of variable mass dynamics as a special discipline of mechanics. He devoted his 160-page master's thesis to exploring a large array of issues relevant to variable mass dynamics - from the derivation of equations of motion to the solution of a series of problems in the field. All of these early investigations of variable mass systems were limited in one way - they were only concerned with the study of the translational motion of

[1] Jiří František August Buqouy (1781-1851); between 1811 and 1815, he was engaged mostly in issues of applied mechanics

[2] Siméon Denis Poisson: "Sur le Mouvement dun System de Corps, en Supposant les Masses Variables"

such systems. The issue of rotational motion of such systems was not addressed until the mid 1940's.

The Second World War brought with it a resurgence of interest and activity in the dynamics of variable mass systems, mostly in connection with rocketry. At this time, translational motion of such systems was relatively well understood and the main focus of research in variable mass dynamics began to shift to the attitude motion of such systems. Some of the scientific giants of this new era include Rosser, Gantmacher and Levin, Rankin, Leitman, and Thomson. These investigators derived several versions of the equations of motion of variable mass systems, and their work gives a great deal of physical insight into the behaviour of rockets in particular.

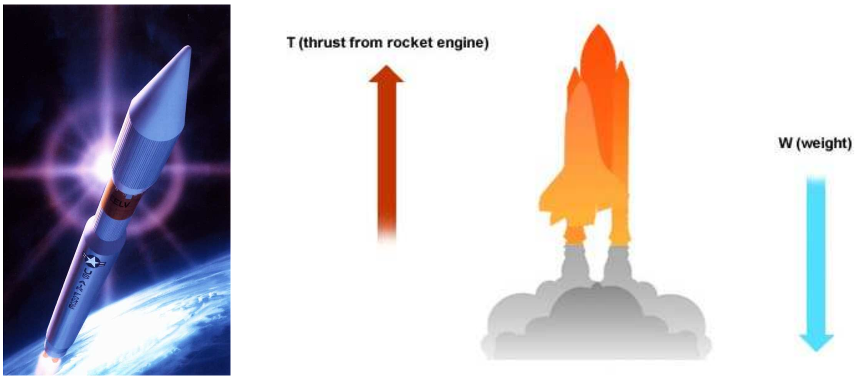
Flaws in current understanding of the dynamics of variable mass systems was brought to light in the early 1980's, when several space missions with upper stages powered by a new class of solid rocket motor were observed to exhibit anomalous behaviour. Unexpected and unexplainable rapid growth in cone angle occurred near the end of the motor burn. This class of motors was the first known to produce such anomaly, and it differed from its predecessors mainly in its much larger size and the existence of a submerged nozzle construction. This problem sparked another flurry of investigations of variable mass systems, for example, Eke, Meyer, Mingori and Yam, Flandro, Cochran and Kang. The derivation of equations of motion for systems with varying mass is not as straightforward as it would normally be for a system of particles and/or rigid bodies of constant mass. The reason is that the basic principles of dynamics, such as NewtonEuler equations and Lagrange's equations are only truly valid when applied to a definite set of particles or rigid bodies. Hence, to study variable mass systems, one can proceed in one of at least two ways. One, is to seek or develop new formalisms that would be valid for variable mass systems in general; another solution is to model variable mass systems in a way that allows them to be viewed as constant mass systems, and thus make them amenable to treatment by existing principles of dynamics.

2. MOVEMENT OF A BODY WITH VARIABLE MASS

Newton's second law

$$\vec{F} = m \cdot \vec{a} = m\frac{d\vec{v}}{dt} = \frac{d\vec{p}}{dt}$$

which states that force is equal to the time rate of change of momentum, is intended to apply to a constant-mass system. However, in many cases the mass variation rate is large, so analysis of the movement of bodies whose mass is changing over time (either due mechanical rejection or acquisition but not in sense of relativity) have great practical importance. The problem of the dynamics of body separation has been known for a very long time. At the end of the nineteenth century and at beginning of the twentieth, Meshchersky laid the foundations of the modern dynamics of a particle with variable mass. "Variable mass systems" are those which expel and/or capture particles during motion. Two kinds of systems are identified, namely, "continuously" particle-ejecting systems, such as rockets, and "discretely" particle-ejecting systems, such as automatic weapons (that fire rounds, one at time). The motion of the continuously mass variable systems is much more investigated due to its application in rocket theory, astronomy, in robotics, in mashinery, etc. For the case of mass continually varying in time, Meshchersky introduced the reactive force



(a) Rocket, Atlas 5 (b) Thrust VS Weight

Figure 1: Reactive motion (a) and thrust (b).

as a function of mass variation and the relative velocity of mass separated or added to the particle.

2. 1. REACTIVE MOTION

Derivation of those basic relations of reactive plant will be illustrated in the following text. However, the presented relations should be considered with caution regarding the assumptions used during the derivation.

If we take for action force $\vec{F}$, the force that ejects mass with velocity $\vec{c}$ for elementary time dt, $\vec{F} = \vec{c}\,\frac{dm}{dt}$, than the reaction force which causing the acceleration $\frac{d\vec{v}}{dt}$ of the mass is $\vec{R} = m\,\frac{d\vec{v}}{dt}$. According to the Newton‘s Third Law, $\vec{R} = -\vec{F}$, so $m\,\frac{d\vec{v}}{dt} = -\vec{c}\,\frac{dm}{dt}$. The vectors $\frac{d\vec{v}}{dt}$ i $\vec{c}$ are collinear, so the previous relation can be written in scalar form $m\,\frac{dv}{dt} = -c\,\frac{dm}{dt}$, where $\frac{dm}{dt}$ is taken to be absolute value ($\frac{dm}{dt} > 0$). More preciously, $\frac{dm}{dt}$ is negative as the mass decreases with time, $\frac{dm}{dt} < 0$ and is called *the mass flow rate.*

On the other hand, if velocity of the body relative to an inertional frame of reference is equal to $\vec{v}$, and if a velocity of the ejected particle relative to the same frame is equal to $\vec{u}$, then $\vec{c} = \vec{u} - \vec{v}$ presents a relative velocity of the ejected particle relative to the moving body. Finally, at progresive motion of the body, this motion can be considered as a motion of a material point only if displacement of the center of mass is negligible during the body mass decreasing. Considering all the above and based on the law of conservation of linear momentum stating that internal forces can not affect a motion of center of mass, a question such as how the reactive force as internal force could couse a body motion can be arised. However, internal forces could move particular parts of the body. Thus, in our case, reactive force[3] as internal force occuring due to mass rejecting, moves the rest of the body upon the rejection. To analyze this question we must consider a system of variable mass, and the process by which it gains velocity as a result of ejecting mass. We shall start by considering a body with velocity $\vec{v}$ and external forces $\vec{F}$, gaining mass at a rate $\dot{m} = dm/dt$. Let us look at the process of gaining a small amount of mass dm. Let $\vec{u}$ be the velocity of dm before it is captured by m, and let $\vec{f}$ represent the average value of the impulsive forces that dm exerts on m during the short interval dt, in which the capturing takes place. By Newton's third law, dm will experience a force $-\vec{f}$, exerted by m, over the same dt.

We can now examine the capture process from the point of view of dm and equate the impulse, $-\vec{f}dt$, to the change in linear momentum of dm, $-\vec{f}dt = dm(\vec{v}+d\vec{v}-\vec{u})$. Here, $\vec{v} + d\vec{v}$ is the velocity of m (and dm) after impact. Analogously, from the point of view of m, we write $\vec{F}dt + \vec{f}dt = m(\vec{v} + d\vec{v}) - m\vec{v} = md\vec{v}$. The term $dmd\vec{v}$ in equation is a higher order term and will disappear when we take limits. The impulse due to the contact force can be eliminated by combining previous equations, $\vec{F}dt - dm(\vec{v} - \vec{u}) = md\vec{v}$ or, dividing through by dt,

$$m\frac{d\vec{v}}{dt} = \vec{F} - (\vec{v} - \vec{u})\frac{dm}{dt} = \vec{F} + (\vec{u} - \vec{v})\frac{dm}{dt} \tag{1}$$

Here, $\vec{u}-\vec{v}$ is the velocity of dm relative to m. This expression is valid when $dm/dt > 0$ (mass gain) and when $dm/dt < 0$ (mass loss). If we compare this expression to the more familiar form of Newton's law for a particle of fixed mass $m\frac{d\vec{v}}{dt} = \vec{F}$, we see that the term $(\vec{u}-\vec{v})dm/dt$ is an additional force on m which is due to the gain (or loss) of mass. Equation (1) can also be written as $\frac{d(m\vec{v})}{dt} = \vec{F} + \vec{c}\frac{dm}{dt}$ where $\vec{c}$ is the velocity of the captured (or expelled) mass relative to the velocity of the mass m. This shows that, for systems involving variable mass, the usual expression stating conservation of linear momentum, $d(m\vec{v})/dt = \vec{F}$, is only applicable when the initial (final) velocity of the captured (expelled) mass, $\vec{c}$, is zero. The behavior of *m(t)* is not an unknown, but is specified according to the characteristics of the rocket. In most cases, $\frac{dm}{dt}$ is a constant and negative. In some cases, the behavior of *m(t)* may be determined by a control system. In any case, it is a given quantity.

2. 2. THE ROCKET EQUATION

The development of rocketry, characterized by the establishment of a theory of interplanetary flight, is inseparably connected with the name of one of the most prominent of Soviet scientists, the founder of rocket dynamics and astronautics, Konstantin Eduardovich Tsiolkovskii (1857-1935). Thoughts of the possibility of conquering space came to Tsiolkovskii very early - when he was only 16 years old. At that time he suggested using the properties of centrifugal force to attain cosmic velocities. Ten years later Tsiolkovskii came to the correct conclusion that the principle of jet propulsion could be used for flight in outer space. This problem was complicated by the fact that since rockets were bodies of variable mass, they could not be treated according to the formulas of classical mechanics, founded on Newton's three laws. It was necessary to

[3] $\vec{F}_r = -\vec{c}\,\frac{dm}{dt} = (\vec{u} - \vec{v})\frac{dm}{dt}$, according to Meshcherskii

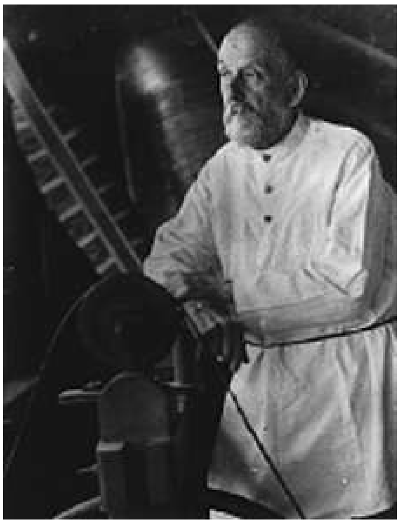


Figure 2: Konstantin Tsiolkovsky, the father of modern rocketry: *"The Earth is the cradle of humanity, but mankind cannot stay in the cradle forever"*.

found a new division of theoretical mechanics - the mechanics of bodies of variable mass. A major role in the resolution of this problem was played by I.V. Meshcherskii. In his master's dissertation "The Dynamics of a Point of Variable Mass, 1897, Meshcherskii deduced the fundamental equation of motion of a point of variable mass. In this paper he took vertical rocket climb as an example, obtaining the equation

$$m\frac{d^2x}{dt^2} = -mg + p - \frac{dm}{dt}W - R(\dot{x}), \qquad Meshcherskii\ equation$$

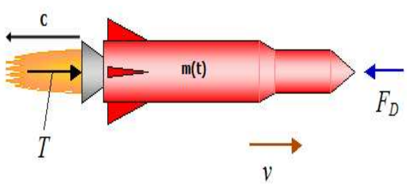
where m is the mass of the rocket; g, the acceleration due to gravity; p, the gas pressure; W, the relative velocity of the burning particles at the moment of their ejection; $\dot{x}$, the velocity of the rocket; and *R(x)*, the air resistance. The x-axis is directed along the ascending vertical. However, because of the fact that at the end of the 19th century rockets had practically no serious use, Meshcherskii limited himself to setting up the problem of rocket motion in its most general form. Tsiolkovskii worked out what was for his time the most complete theory of rocket motion with a calculation of the mass change. As early as 1897 he deduced his now well-known formula, establishing the analytical relation between the rocket velocity, the exaust velocity of the gas particles, the mass of the rocket, and the mass of the charge consumed:

$$V = V_1 \ln\left(1 + \frac{M_2}{M_1}\right), \qquad Tsiolkovkii\ formula$$

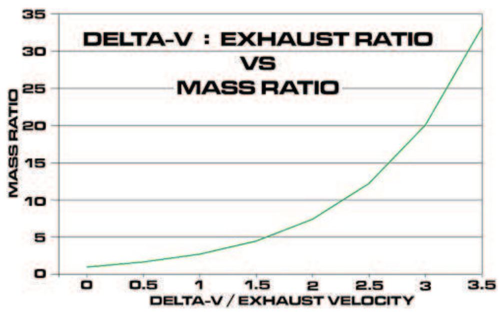
where V is the velocity of the rocket; V_1, the relative exhaust velocity of the gas particles; M_1, the mass of the rocket without explosive charge; and M_2, the mass of the charge. It is evident from Tsiolkovskii's formula that the velocity of rocket in free space depends only upon the exhaust velocity of the gas particles and the relation between the mass of the charge and the mass of the rocket. Also, Tsiolkovskii's formula showed that to increase the velocity of rockets, exhaust velocity of the gas particles had to be increased and the relative (not the absolute) fuel supply enlarged.

Rocket physics is basically the application of Newton's Laws to a system with variable mass. A rocket has variable mass because its mass decreases over time, as a result of its fuel (propellant) burning off. A rocket obtains thrust by the principle of action and reaction (Newton's third law). As the rocket propellant ignites, it experiences a very large acceleration and exits the back of the rocket (as exhaust) at a very high velocity. This backwards acceleration of the exhaust exerts a "push" force on the rocket in the opposite direction, causing the rocket to accelerate forward. This is the essential principle behind the physics of rockets, and how rockets work.

The equations of motion of a rocket will be derived next. To find the equations of motion, apply the principle of impulse and momentum to the "system", consisting of rocket and exhaust. We consider a rocket of mass m, moving at velocity $\vec{v}$ and subject to external forces $\vec{F}$ (typically gravity and drag). The rocket mass changes at a rate $\dot{m} = dm/dt$, with a velocity vector $\vec{c}$ relative to the rocket. We shall assume that the magnitude of $\vec{c}$ is constant.



(a) The "system" consists of rocket and exhaust



(b) High mass ratios mean ridiculously high propellant loads

Figure 3: Reactive force (a) and mass ratios (b).

The velocity of the gas observed from a stationary frame will be $\vec{u} = \vec{v} + \vec{c}$. In this frame, $\vec{c}$ is a vector aligned along the flight path in a negative direction, $\vec{c} = -c\vec{\tau}$, where $\vec{\tau}$ is the unit direction along the flight path. Thus,

$$m\frac{d\vec{v}}{dt} = \vec{F} + \vec{c}\frac{dm}{dt} \tag{2}$$

The term $\vec{T} = \vec{c}\ dm/dt$ is called the *thrust* of the rocket and can be interpreted as an additional force on the rocket due to the gas expulsion. Equation (9) is a vector equation which can be projected along the direction of $\vec{v}$ (tangent to the path). Thus,

$$m\frac{dv}{dt} = F_t - c\frac{dm}{dt} = F_t + T \tag{3}$$

where F_t is the tangential component of $\vec{F}$, v and c are the magnitudes of $\vec{v}$ and $\vec{c}$ respectively, and we have assumed that $\vec{c}$ is parallel and has opposite direction to $\vec{v}$. The magnitude of the thrust is $T = -c\dot{m}$. Note that for a rocket, $\dot{m}$ will be negative (mass is lost). If the force F_t is known, this equation can be integrated in time to yield an expression for the velocity as a function of time. Let us consider some simple cases:

No External Forces: $F_t = 0$
If gravity and drag effects are neglected, we have, $m\frac{dv}{dt} = -c\frac{dm}{dt}$ or, integrating between an initial time t_0, and a final time t,

$$\Delta v = v - v_0 = -c(\ln m_f - \ln m_0) = -c \ln \frac{m_f}{m_0} = c \ln \frac{m_0}{m_f} \quad (4)$$

Alternatively, this expression can be cast as the well known rocket equation, $m = m_0\ e^{-\Delta v/c}$ which gives the mass of the rocket at a time t, as a function of the initial mass m_0, Δv, and c. The mass of the propellant, $m_{propellant}$, is given by,

$$m_{propellant} = m_0 - m = m_0 \left(1 - e^{-\Delta v/c}\right)$$

Gravity: $F_t = -mg$
A constant gravitational field acting in the opposite direction to the velocity vector can be easily incorporated. In this case, equation (3) becomes, $m\frac{dv}{dt} = -mg - c\frac{dm}{dt}$ which can be integrated to give

$$v = v_0 - c \ln \frac{m}{m_0} - gt = v_0 - c \ln \frac{m}{m_0} - g\frac{m - m_0}{\dot{m}} \quad (5)$$

This solution assumes that $c\dot{m} > m_0 g$ at t=0. If this is not true, the rocket will sit on the pad, burning fuel until the remaining mass satisfies this requirement.

CONCLUSION:

Variable mass systems mechanics is the branch of Mechanics within systems which show variation in the amount of mass to them associated are considered. There is an inherent issue in the treatment of this kind of system. Principles of Mechanics were originally conceived for constant mass systems. Thus, the prime form of the representative equations of such principles cannot be straightly applied on variable mass systems. Nevertheless, the motion of a rocket is a nice example of a system with a variable mass in which conservation of linear momentum can be applied.
Developments in the field of variable mass systems have led to the development of important scientific instruments (gyroscopes, etc.) and to the concept of spin stabilization of modern spacecraft. The finite limits of electrical energy, thrust, and propellant on a vehicle are the driving forces that demand efficient and accurate equipment to perform the functions of steering and navigation while keeping the vehicle attitude stabilized. Added to these restrictions are the needs for minimum volume and mass.

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