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ITERATION FACTORS METHOD FOR MULTILEVEL RADIATIVE TRANSFER: CONVERGENCE PROPERTIES

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Abstract. We present the Iteration Factors Method (IFM) developed to solve the multilevel line transfer problem and show its excellent convergence properties in the well-known five-level Ca II test case.

1. INTRODUCTION

The solution of radiative transfer (RT) is an important and inevitable step in many astrophysical problems. The NLTE radiative transfer problems are difficult to deal with due to the non-local coupling of the radiation field and the state of gas. In comparison with the two-level atom line formation problem, where the line source function can be expressed explicitly in terms of the line radiation field, the multilevel atom problem has an additional difficulty, arising from the non-linear coupling of the atomic level populations and the radiation field intensities in the corresponding line transitions. Hence, an iterative method is required for solving the radiative transfer (RT) and the statistical equilibrium (SE) equations simultaneously.

A broad class of efficient methods – the so-called Accelerated Lambda Iteration (ALI) methods has been developed in the last few decades to cope with the problem (for an overview see Hubeny, 2003). In this paper we present an alternative method – the Iteration Factors Method (IFM) that provides an extremely fast and stable convergence to the exact solution. The IFM was developed for the two-level atom line transfer by Atanacković-Vukmanović and Simonneau (1994, Paper I) and extended to the multilevel atom line formation problem by Kuzmanovska-Barandovska and Atanacković (2010, Paper II). Here, we summarize the basic idea of the method and demonstrate its good convergence properties by solving the well-known 5-level CaII test problem.

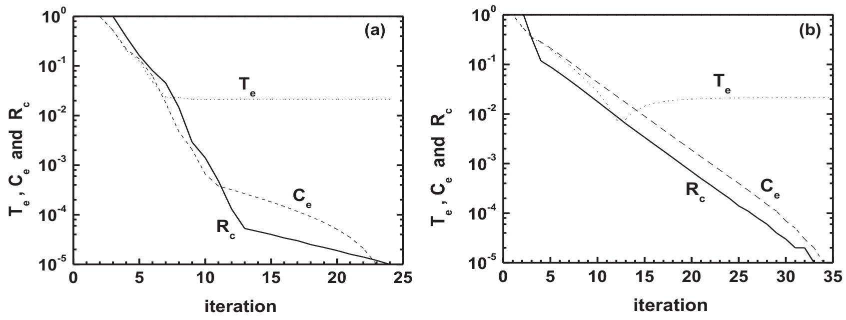

Figure 1: Variation of T_e, C_e and R_c with iteration number for the F2-L solution using two relaxation parameters: (a) r=0.25 and (b) r=0.75.

Table 1: Number of iterations needed by the four IFM procedures to satisfy the given convergence criteria.

Convergence criteria	F1-L	F1-M	F2-L	F2-M
$C_e < 10^{-2}$	7	10	8	9
$R_c < 0.1T_e(\infty)$	12	17	10	20

2. ITERATION FACTORS METHOD

The Iteration Factors Method enables very fast and exact solution of the RT problems by means of quasi-invariant functions, the so-called iteration factors, within a simple sequential iterative procedure. In the first part of each iteration, from the formal solution of the RT equation (with a given LTE source function in the first iteration, or the one known from the previous iteration step) for each line transition, the radiation field intensities, their angle and frequency integrated moments and the ratios of these moments (iteration factors) are computed. The factors are then used in the second part of iteration to close the RT equation moments, non-linearly coupled with the SE equations. We apply two approaches to solve the non-linear problem: linearization of the relevant equations, and the modification of the SE equations to make them linear. In the first approach we linearize the equations by expanding all the relevant variables to the first order. The changes in the emission and absorption coefficients of the linearized RT equations are expressed through the changes in the level populations; these, in turn, are expressed using the linearized SE equations in terms of the corrections of the mean line radiation field intensities δJ_φ. In this way the solutions of the resulting system satisfy both sets of equations, RT and SE. In the second approach we write the SE equations in a linear form assuming that the level populations in the line-opacity-like terms are known from the previous iteration. The rate equations modified in such a way are substituted in the RT equation moments resulting in the system for J_φ only. In both approaches, the RT equation moments are closed by the iterations factors, enabling rapid convergence to the exact solution.

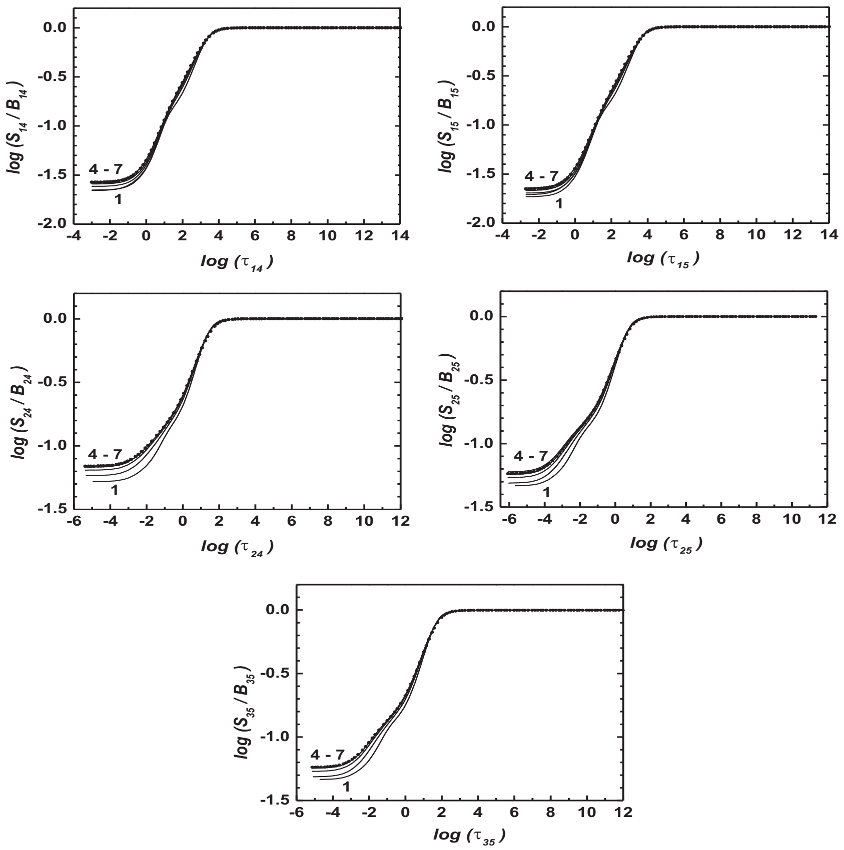

Figure 2: The convergence of the source function for five line transitions obtained with the second family of iteration factors and the modified SE equations (F2-M).

3. CONVERGENCE PROPERTIES

In order to demonstrate the efficiency and accuracy of the method, here we present the results obtained for the 5-level Ca II line formation in a plane parallel, semi-infinite and constant property medium – the test problem proposed by Avrett (1968). Two families of iteration factors (F1 and F2) defined in Paper I and two approaches to the solution of the non-linear coupling of RT and SE equations (linearization (L) and the modification of the SE equations (M)), described in more details in Paper II, are used for its solution.

We analyzed the convergence behavior of all four procedures (F1-L, F1-M, F2-L and F2-M) by computing the following quantities at each iteration step i:

$$R_c^i = |\frac{S^i - S^{i-1}}{S^i}|_{\max}\ , \quad C_e^i = |\frac{S^i - S^\infty}{S^\infty}|_{\max}\ , \quad T_e^i = |\frac{S^i - S^\infty_{\rm FBILI}}{S^\infty_{\rm FBILI}}|_{\max}\ , \qquad (1)$$

where R_c is the maximum relative change of the source function between two successive iterations, C_e is the maximum relative convergence error and T_e is the maximum relative true error. Here, S^{∞} and $S^{\infty}_{\mathrm{FBILI}}$, respectively, are the fully converged solutions obtained with the Iteration Factors method and the Forth-and-Back Implicit Lambda Iteration (FBILI) method developed by Atanacković-Vukmanović, Crivellari and Simonneau (1997). The FBILI solutions are used as the reference ones because we could straightforwardly compare the results obtained by both methods. In all the computations the same discretization in optical depth (6 points per decade) is used.

In Fig. 1 we display the variation of T_e, C_e and R_c with the iteration number for the solution obtained using more refined (second) family of iteration factors and the linearization (F2-L). Two values of under-relaxation parameter r ($0 < r < 1$, see Paper II) are used. The convergence is extremely fast, reaching R_c of about 10^{-5} in 24 iterations with $r = 0.25$ and in 35 iterations with $r = 0.75$. The maximum relative true error reaches its asymptotic value $T_e(\infty)$ of 2% after 7 and 16 iterations, respectively. For all four solutions, in Table 1 we give the number of iterations needed to satisfy the usual convergence criteria $C_e \leq 10^{-2}$ and $R_c \leq 0.1T_e(\infty)$. We see that the convergence in all the cases is rapid. The solutions that differ by 1-3% from the FBILI solutions are obtained in only 10-20 iterations.

The behavior of the source function over iterations for all five spectral lines is shown in Fig. 2. Here, we displayed the solutions obtained with the second family of factors and the modified SE equations (F2-M). The usual stopping criterion $R_c < 1\%$ is satisfied in only 7 iterations. Let us note that the good thermalization length is achieved already in the first iteration, while the true accuracy of 2% is reached after only a few iterations.

The Iteration Factors Method has proven to be a fast and accurate method with the convergence properties comparable to those of the most efficient methods developed to solve the multilevel line transfer problems.

Acknowledgments

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