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EQUILIBRIA OF SEELIGER'S PROBLEM. ANALYTIC APPROACH

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Abstract. We offer a deeper insight into the two-body problem associated to Seeliger's potential. We resort to McGehee-type transformations to write the motion equations and the integrals of energy and angular momentum. Then we search for equilibria, considering the whole interplay among the parameter A of the field, the constant K of the field, and the integration constants of energy h and angular momentum C. We find the number of equilibria for each such situation.

1. INTRODUCTION

At the end of the 19th century, the German astronomer Hugo von Seeliger formulated a "philosophical astonishment" concerning the stability of the whole Universe: "Why are we not crashed under the infinite pressure yielded by the infinite number of stars of the Universe, as predicted by the Newtonian Mechanics?" (see Ionescu-Pallas 2003). Such an "astonishment" comes from the fact that, for a uniform distribution of masses, Laplace-Poisson's equation has no finite solution corresponding to a nonempty Universe (see the so-called Newtonian cosmology).

In classical mechanics, the only possibility to overcome this difficulty is to generalize Laplace-Poisson's equation by adding a term analogous to the cosmological constant from relativity. Following this way, Seeliger (1895, 1896) proposed a new theory of gravitation that obviously entailed a revision of the classical Newtonian mechanics. Special Relativity Theory and General Relativity Theory applied to celestial mechanics made Seeliger's model fall into oblivion. Nevertheless, connections between Seeliger's and Einstein's theories, especially as to fundamental constants, were pointed out (Pauli 1921; Hubble 1936; Jones et al. 1956; Tifft 1995).

The gravitational potential characteristic to Seeligeer's theory reads:

$$U(\mathbf{r}) = \frac{A}{r} e^{-Kr}, \tag{1}$$

where $r = |\mathbf{r}| =$ the distance between two pointlike masses m_1 and m_2, $A = Gm_1 m_2 > 0$ ($G =$ Newtonian constant of gravitation), K is a positive constant of order $10^{-28}\mathrm{cm}^{-1}$.

Mioc and Rusu (2006) performed a first insight into the two-body problem associated to Seeliger's model (collision, escape, symmetries). In this paper we go deeper into Seeliger's problem, searching for equilibria.

2. BASIC EQUATIONS IN CONFIGURATION-MOMENTUM COORDINATES

Seeliger's potential is central, hence the two-body problem can be reduced to a central-force problem. The motion is planar, and is ruled by the Hamiltonian

$$H(\mathbf{q}, \mathbf{p}) = \frac{|\mathbf{p}|^2}{2} - \frac{A}{|\mathbf{q}| e^{K|\mathbf{q}|}}. \tag{2}$$

Here $\mathbf{q} = (q_1, q_2) \in \mathbf{R}^2 \setminus \{(0,0)\}$ and $\mathbf{p}(= \dot{\mathbf{q}}) = (p_1, p_2) \in \mathbf{R}^2$ stand, respectively, for the position (configuration) vector and the momentum vector of a unit-mass particle (in conveniently chosen units) with respect to the field source (centre).

The problem admits two integrals in involution: energy and angular momentum:

$$H(\mathbf{q}, \mathbf{p}) = h = \text{ constant}, \tag{3}$$

$$L(\mathbf{q}, \mathbf{p}) = q_1 p_2 - q_2 p_1 = C = \text{ constant}. \tag{4}$$

Explicitly, the equations of motion read

$$\dot{\mathbf{q}} = \mathbf{p}, \quad \dot{\mathbf{p}} = -A e^{-K|\mathbf{q}|} \Big(\frac{K}{|\mathbf{q}|^2} + \frac{1}{|\mathbf{q}|^3} \Big) \mathbf{q}. \tag{5}$$

3. BASIC EQUATIONS IN McGEHEE-TYPE COORDINATES

The potential, the motion equations and the energy integral have an isolated singularity at the origin $\mathbf{q} = (0,0)$, which corresponds to a collision particle-centre. To regularize equations (5), we resort to the McGehee-type (1974) transformations:

$$r = |\mathbf{q}|, \quad \theta = \arctan\Big(\frac{q_2}{q_1}\Big), \quad u = \dot{r} = \frac{q_1 p_1 + q_2 p_2}{|\mathbf{q}|}, \quad v = r\dot{\theta} = \frac{q_1 p_2 - q_2 p_1}{|\mathbf{q}|},$$

$$x = r^{\frac{1}{2}} u, \quad y = r^{\frac{1}{2}} v, \quad ds = r^{-\frac{3}{2}} dt.$$

So, writing $(\cdot)' = d(\cdot)/ds$, the equations of motion (5) become

$$r' = rx, \quad \theta' = y, \quad x' = \frac{x^2}{2} + y^2 - A e^{-Kr}(Kr + 1), \quad y' = -\frac{xy}{2}. \tag{6}$$

A straightforward computation leads to the new expressions of the first integrals:

$$\frac{x^2 + y^2}{2} - A e^{-Kr} = hr, \tag{7}$$

$$ry^2 = C^2. \tag{8}$$

4. EQUILIBRIA

We shall tackle further down the equilibria of the problem. We shall consider only those equilibria for which $0 < r < +\infty$ (see Mioc and Rusu 2006). From the integrals (7) and (8) we immediately get

$$\frac{x^2}{2} = -\frac{C^2}{2r} + Ae^{-Kr} + hr. \tag{9}$$

The left-hand side is nonnegative. So, for every fixed values of h and C, the regions where real motion is allowed are featured by

$$-\frac{C^2}{2r} + Ae^{-Kr} + hr \geq 0.$$

We define the effective potential as

$$V_{\text{eff}}(r;C) = \frac{C^2}{2r^2} - \frac{A}{re^{Kr}}.$$

Then, for a fixed energy level h, the regions of possible motion are given by the condition $V_{\text{eff}}(r;C) \leq h$.

Given the exponential character of the potential (and effective or amended potential), we think that this condition must be verified numerically. This will be performed in our next attempt to tackle the local and global flow of the problem.

Coming back to equilibria, θ does not appear explicitly in either the right-hand side of motion equations or first integrals. So, we may discard the equation corresponding to θ in (6). Now we are interested only in the system formed by

$$r' = rx, \quad x' = \frac{x^2}{2} + y^2 - Ae^{-Kr}(Kr+1), \quad y' = -\frac{xy}{2}, \tag{10}$$

and the integrals (7) and (8). With this, the equilibria of the problem are of the form $(r, x, y) = (\widetilde{r} > 0, \widetilde{x} = 0, \widetilde{y})$ and must be solutions of the system

$$\widetilde{y}^2 - \frac{A(K\widetilde{r}+1)}{e^{K\widetilde{r}}} = 0, \tag{11}$$

$$\frac{\widetilde{y}^2}{2} - \frac{A}{e^{K\widetilde{r}}} = h\widetilde{r}, \tag{12}$$

$$\widetilde{r}\widetilde{y}^2 = C^2. \tag{13}$$

Eliminating $\widetilde{y}$ between (11)-(13), compatibility relations must be fulfilled:

$$\frac{A(K\widetilde{r}+1)}{e^{K\widetilde{r}}} = \frac{2A}{e^{K\widetilde{r}}} + 2h\widetilde{r} = \frac{C^2}{\widetilde{r}}. \tag{14}$$

In this way, the function given by (9):

$$x^2 = f(r) = -\frac{C^2}{r} + 2Ae^{-Kr} + 2hr$$

becomes for equilibria:

$$f(\widetilde{r}) = \frac{A(1 - K\widetilde{r})}{e^{K\widetilde{r}}} + 2h\widetilde{r}. \tag{15}$$

First consider the values of the angular momentum. For $C = 0$ (rectilinear motion), by (13), $\widetilde{y} = 0$. But, since $K > 0$, (11) leads to a contradiction. Therefore, there are no equilibria (outside the collision manifold; see Mioc and Rusu 2006) with $C = 0$.

For $C \neq 0$, we plug (13) into (11) and get

$$\frac{C^2}{\widetilde{r}} - \frac{A(K\widetilde{r} + 1)}{e^{K\widetilde{r}}} = 0.$$

By (9), since $\widetilde{x} = 0$, it follows that $C^2/(2\widetilde{r}) - A/e^{K\widetilde{r}} = h\widetilde{r}$. Eliminating the terms which contain exponentials between the last equations, and taking into account the fact that $\widetilde{r} \neq 0$, after a straightforward calculation we obtain:

$$2hK\widetilde{r}^3 + 2h\widetilde{r}^2 - C^2K\widetilde{r} + C^2 = 0. \tag{16}$$

In this manner, we removed both the exponentials and the parameter A. Equation (16) constitutes the basis for our searching for equilibria. We shall discuss its positive roots resorting to Descartes' rule of signs and Rolle's sequence.

There are several cases, and we investigate them according to the energy level.

4.1. **Negative energy**. For $h < 0$, (16) has only one change of sign; this means exactly one positive root. So, the problem admits only one equilibrium. Its location from the centre (analytic expression of corresponding $\widetilde{r}$) can be found by the lengthy solution of (16).

This analysis can be refined by denoting the left-hand side of (16) by

$$g(\widetilde{r}) = 2hK\widetilde{r}^3 + 2h\widetilde{r}^2 - C^2K\widetilde{r} + C^2.$$

So,

$$g'(\widetilde{r}) = 6hK\widetilde{r}^2 + 4h\widetilde{r} - C^2K.$$

The equation $g'(\widetilde{r}) = 0$ has the roots

$$\widetilde{r} = \frac{-2h \pm \sqrt{\Delta}}{6hK}, \qquad \Delta = 4h^2 + 6hK^2C^2.$$

We distinguish three subcases: h smaller than/equal to/greater than $-3K^2C^2/2$. They respectively entail $\Delta > 0$, $\Delta = 0$, $\Delta < 0$. Using the fact that g is obviously strictly decreasing for $h < 0$, as well as Rolle's sequence, we obtain in all three cases the same result: a single positive root of (16), hence only one equilibrium.

4.2. **Zero energy**. For $h = 0$, (16) also has only one change of sign; this means exactly one positive root. So, in this case the problem also admits only one equilibrium. Its location from the centre (analytic expression of corresponding $\widetilde{r}$) is now simple. From (16) it results immediately:

$$\widetilde{r} = \frac{1}{K}.$$

(Taking into account the order of magnitude of K, this equilibrium is very far from the centre.)

4.3. **Positive energy**. For $h > 0$, we have two changes of sign, therefore two positive roots, or no positive root at all.

Here we distinguish three situations. We shall discuss the roots of (16), as usual, by letting C vary.

If $C^2 (\neq 0)$ is very small, the graph of $g(\widetilde{r})$ does not intersect the $\widetilde{r}$-axis. This means no positive root at all, hence no equilibrium.

Let us C^2 increase. There appears a critical value (C^2_{cr}) for which the graph of $g(\widetilde{r})$ has a double zero. This means one positive double root, hence one equilibrium.

Let us C^2 increase more ($C^2 > C^2_{\mathrm{cr}}$). The graph of $g(\widetilde{r})$ has two simple zeroes. This means two different positive roots, hence two equilibria.

Here, too, the analysis can be refined, avoiding graphs. Keeping the notation used in the negative-energy case, we easily see that the equation $g'(\widetilde{r}) = 0$ admits only one positive root:

$$\widetilde{r} = \frac{-2h + \sqrt{\Delta}}{6hK}.$$

With this value we get:

$$g(\widetilde{r}) = \frac{4h^3 + 51h^2K^2C^2 - \sqrt{\Delta}(2h^2 + 3hK^2C^2)}{27h^2K^2}.$$

The sign of $g(\widetilde{r})$ is given by the sign of

$$E(h) = 336h^2 + 2493hK^2C^2 - 54K^4C^4.$$

The equation $E(h) = 0$ has only one positive root, $h^+ \cong 14.5137K^2C^2$. One immediately sees that $E(h) < 0$, $E(h) = 0$, $E(h) > 0$ for $h \in (0, h^+)$, $h = h^+$, $h \in (h^+, +\infty)$, respectively. Now, using Rolle's sequence, a straightforward calculation shows that for $h \in (0, h^+)$, $h = h^+$, $h \in (h^+, +\infty)$ the equation $g(\widetilde{r}) = 0$ has, respectively two postive roots, one positive double root, and no positive root. In physical terms, this respectively means: two equilibria, one equilibrium, and no equilibrium at all. This confirms the results that used graphs.

Observe that this last purely analytic approach handled the constant of energy instead of the angular momentum constant. Also observe that this approach avoids the geometric representation and offers a more refined analysis of the possible cases of interplay among the constants of the field and the integration constants.

Of course, in this case of positive energy, the locations from the centre (analytic expression of corresponding $\widetilde{r}$) can be found, as in the negative-energy case, by the lengthy solution of (16).

5. CONCLUDING REMARKS

Since the potential is exponential, the problem of finding equilibria is complicated. Nevertheless, we managed to remove the exponential, obtaining a starting algebraic equation, which did no longer contain the parameter A.

We found the number of equilibria for the whole interplay between C and h. We have pointed out all possible cases. The location of the equilibrium was found only in the zero-energy case. In the other cases the analytic calculations are very intricate. The nature of equilibria will be clarified and substantiated elsewhere.

Seeliger's problem admits equilibria even in the case of nonnegative-energy. Such situations are not encountered in the two-body problem in the Newtonian field, where equilibria occur only for negative-energy levels.

Even if the results of this paper are based mainly on analytical methods, for the next step (tackling the local and global flow), the numerical methods (supported by geometrical methods) will be the main tool.

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