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FULFILLMENT OF THE CONDITIONS FOR APPLICATION OF THE NEKHOROSHEV THEOREM USING EXTENDED HAMILTONIAN

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Abstract. This paper represents a part of the project devoted to application of the theorem of Nekhoroshev to asteroids. Spectral formulation of the Nekhoroshev theorem has been applied by Guzzo et al. (2002) to selected asteroids with regular motion in the region of Koronis family and with chaotic motion in the region of Veritas family. In the paper by Pavlović and Guzzo (2008) the fulfilment of conditions for application of the Nekhoroshev theorem in the same regions have been checked by using an integrable Hamiltonian which consisted of Kepler's and Kozai's terms. Here we expand the Hamiltonian with the most important terms linear in e' and i' of Jupiter and Saturn. With such expanded Hamiltonian we checked the fulfilment of conditions of convexity, quasi-convexity and 3-jet non-degeneracy in the regions of the two families and in the vicinity of secular resonances. The obtained results are compared with results obtained previously.

1. INTRODUCTION

The application of the theorem of Nekhoroshev to assess the stability of motion of asteroids over the exponentially long time spans is particularly important in terms of the study of their origin and dynamical evolution. The theorem in its spectral formulation has for the first time been successfully used for this purpose by Guzzo et al. (2002). They studied the spectra of a number of bodies with different dynamical properties and concluded that these spectra do indeed reveal the basic properties of motion, ranging from ostensibly stable to strongly chaotic. It is, however, well known that the Nekhoroshev theorem provides an exponential stability estimate for quasi-integrable non-degenerate Hamiltonian systems (which is not the case with the asteroid Hamiltonian). Moreover, Nekhoroshev (1977) proved this stability result only for the Hamiltonians that are analytic perturbations of steep functions. It was thus necessary to verify that the conditions for the application of the Nekhoroshev theorem are fulfilled in the regions of the asteroid main belt of interest; this was first done by Pavlović and Guzzo (2008) and later improved by Pavlović and Knežević (2008).

Pavlović and Guzzo (2008) checked the fulfillment of the conditions for application of the Nekhoroshev theorem by using the Kozai's Hamiltonian which is known to be only a rough approximation of the real dynamics. Kozai's approximation lacks terms of higher degrees and orders, which in many cases are more important than those actually accounted for. In the previous paper (Pavlović and Knežević 2008) we managed to partly overcome this problem by introducing additional terms in the Hamiltonian, while keeping it still integrable and simple enough to allow for the cumbersome computations involved with the checking of conditions. We closely followed, up to the point, the procedure by Morbidelli (1993) to compute asteroid resonant proper elements, thus making use of Kozai's terms complemented by the terms linear in planetary eccentricities and inclinations. In the present paper we provide additional results along the same line of investigation, and we offer some new results regarding the asteroids located in or near the strong linear secular resonances (included in the dynamical model through the newly added terms).

The condition of steepness involved with the proof of the Nekhoroshev theorem has been originally written in an implicit form difficult to use in practical applications. One must, therefore, use some stronger sufficient conditions, that can be explicitly written in terms of the derivatives of the Hamiltonian of the problem. These conditions are convexity, quasi-convexity and 3-jet non-degeneracy (Nekhoroshev 1977), employed e.g. by Benettin et al. (1998) to study the stability of the Lagrangian points.

A flow chart and a detailed explanation of the algorithm to check the conditions of convexity, quasi-convexity and 3-jet non-degeneracy are given in Knežević and Pavlović (2008, Fig. 4). For the sake of completeness, we again provide here only a brief description of the procedure. Once the integrable Hamiltonian is known and the corresponding Hessian derived, one proceeds with the check of the signs of its eigenvalues. If all the three signs are the same (more precisely, if the Hessian is either positive or negative definite), the Hamiltonian is *convex*. If one of the signs differ, the restriction of the Hessian to the plane orthogonal to the frequency vector ω is considered; if it is either positive or negative definite, that is, if the signs of its two eigenvalues are the same, the Hamiltonian is *quasi-convex*. If the signs differ one must first compute vectors $u^{\pm}$ in the plane orthogonal to the frequency vector ω defined by an additional condition, and then check the fulfillment of the *3-jet* condition itself. This involves computation of the third derivatives of the Hamiltonian, which in the case of the asteroid Hamiltonian must be done by means of the semi-numerical techniques, extending the techniques introduced by Henrard (1990), Henrard and Lemaitre (1986) and Lemaitre and Morbidelli (1994).

If none of the above conditions are fulfilled, we conclude that the Nekhoroshev theorem cannot be applied for a given asteroid. Since in practice it is difficult to get an exact zero as the result of the above procedure, to distinguish between fulfillment of the 3-jet condition and no condition at all we used as the threshold value an arbitrary small value of 0.01 (the value used by Lamaitre and Morbidelli (1994) as a resonance width criterion). Increasing this value would increase also the number of "failures", that is of asteroids for which conditions are not fulfilled, and vice versa.

2. HAMILTONIAN AND ITS DERIVATIVES

In the present paper we used an extended Hamiltonian given by:

$$h = h_0 + \varepsilon \mathcal{K}_0 + \varepsilon \mathcal{K}_1 \tag{1}$$

where $\mathcal{K}_0$ is Kozai's part, and $\mathcal{K}_1$ is the perturbation of Kozai's part linear in planetary e', i' (Pavlović and Knežević 2008).

Morbidelli (1993) uses the Fourier expansion of $\mathcal{K}_1$:

$$\begin{aligned}
\mathcal{K}_1 \quad = \quad & \sum_{k=-n}^{n} \mathcal{K}_{1,k}^{1}(\Lambda, J, Z)\cos(2k\psi - z - g_5 t - \lambda_5^0) + \\
& \sum_{k=-n}^{n} \mathcal{K}_{1,k}^{2}(\Lambda, J, Z)\cos(2k\psi - z - g_6 t - \lambda_6^0) + \\
& \sum_{k=-n}^{n} \mathcal{K}_{1,k}^{3}(\Lambda, J, Z)\cos((2k+1)\psi - z - s_6 t - \mu_6^0)
\end{aligned} \tag{2}$$

to write the generating function of the transformation:

$$\{W_1, \mathcal{K}_0\} + \mathcal{K}_1 = 0 \tag{3}$$

in an explicit form:

$$\begin{aligned}
W_1 \quad = \quad & \sum_{k=-n}^{n} -\frac{\mathcal{K}_{1,k}^{1}(\Lambda, J, Z)}{k\nu_\psi - \nu_z - g_5}\sin(2k\psi - z - g_5 t - \lambda_5^0) + \\
& \sum_{k=-n}^{n} -\frac{\mathcal{K}_{1,k}^{2}(\Lambda, J, Z)}{k\nu_\psi - \nu_z - g_6}\sin(2k\psi - z - g_6 t - \lambda_6^0) + \\
& \sum_{k=-n}^{n} -\frac{\mathcal{K}_{1,k}^{3}(\Lambda, J, Z)}{(k+1)\nu_\psi - \nu_z - s_6}\sin((2k+1)\psi - z - s_6 t - \mu_6^0)
\end{aligned}$$

allowing to analyze the effect of each perturbing term separately.

As explained above, to check the fulfillment of conditions for application of the Nekhoroshev theorem, we need to compute the corresponding Hessian, that is the matrix of the second derivatives over the actions of the Hamiltonian (1) averaged over the angles; even the third derivatives of the Hamiltonian are needed to check the 3-jet condition. The computation of the derivatives of Kozai's Hamiltonian $\mathcal{K}_0$ is described in Pavlović and Guzzo (2008), thus we discuss here only the procedure of transformation of the perturbation $\mathcal{K}_1$. We make use of the first order semi-numerical method of Henrard (1990) to average $\mathcal{K}_1$ over the slow angles ψ and z and get the Hamiltonian independent of the angles:

$$\bar{h}(\overline{\Lambda}, \overline{J}, \overline{Z}) = h_0(\overline{\Lambda}) + \varepsilon\bar{\mathcal{K}}_0(\overline{\Lambda}, \overline{J}, \overline{Z}) + \varepsilon\bar{\mathcal{K}}_1(\overline{\Lambda}, \overline{J}, \overline{Z}). \tag{4}$$

Relations between the old and new action-angle variables are given by the implicit relations:

$$\begin{aligned}
\Lambda &= \overline{\Lambda}, & \lambda &= \overline{\lambda} - \frac{\partial W_1}{\partial \overline{\Lambda}}(\overline{\Lambda}, \overline{J}, \overline{Z}, \overline{\psi}, \overline{z}), \\
J &= \overline{J} + \frac{\partial W_1}{\partial \tilde{\psi}}(\overline{\Lambda}, \overline{J}, \overline{Z}, \overline{\psi}, \overline{z}), & \psi &= \overline{\psi} - \frac{\partial W_1}{\partial \tilde{J}}(\overline{\Lambda}, \overline{J}, \overline{Z}, \overline{\psi}, \overline{z}), \\
Z &= \overline{Z} + \frac{\partial W_1}{\partial \tilde{z}}(\overline{\Lambda}, \overline{J}, \overline{Z}, \overline{\psi}, \overline{z}), & z &= \overline{z} - \frac{\partial W_1}{\partial \tilde{Z}}(\overline{\Lambda}, \overline{J}, \overline{Z}, \overline{\psi}, \overline{z}),
\end{aligned}$$

which are solved iteratively (Lemaitre and Morbidelli 1994).

Derivatives of $\bar{\mathcal{K}}_1$ over actions $(\overline{\Lambda}, \overline{J}, \overline{Z}) = (J_1, J_2, J_3)$ are obtained starting from:

$$\bar{\mathcal{K}}_1 = \frac{1}{T} \int_0^T K_1(t) dt, \tag{5}$$

As, in fact, $K_1(t) = K_1(\Lambda, q(t), Q(t), z(t), Z)$, we compute it here by replacing $q(t)$ and $Q(t)$ with the solutions of the equations of motion of Kozai's Hamiltonian $K_0(\Lambda, q, Q, P)$, while $z(t)$ is computed from:

$$z(t) = z_0 - \nu_z t + \int_0^t \frac{\partial K_0}{\partial P}(\Lambda, q(t'), Q(t'), P) dt'. \tag{6}$$

Note that T is the period of the unperturbed orbit of Kozai's Hamiltonian, and $\nu_z = \partial K_0 / \partial Z$.

By differentiating (5) we find:

$$\begin{aligned} \frac{\partial \bar{\mathcal{K}}_1}{\partial J_i} &= \left[(K_1)_{t=T} - \frac{\bar{\mathcal{K}}_1}{T} \right] \frac{\partial T}{\partial J_i} \\ &+ \frac{1}{T} \int_0^T \left[\frac{\partial K_1}{\partial q} \frac{\partial q}{\partial J_i} + \frac{\partial K_1}{\partial Q} \frac{\partial Q}{\partial J_i} + \frac{\partial K_1}{\partial z} \frac{\partial z}{\partial J_i} + \frac{\partial K_1}{\partial J_i} \right] dt, \end{aligned} \tag{7}$$

where $(\frac{\partial q}{\partial J_i}, \frac{\partial Q}{\partial J_i}, \frac{\partial z}{\partial J_i})$ is the solution of the set of variational equations.

The second and the third derivatives over the actions are found by repeating the above procedure, resulting, however, in rather cumbersome expressions.

Once the derivatives are computed, we form the Hessian and compute the corresponding sums of the third derivatives, passing these subsequently as input into the procedure of checking the conditions of the previous Section.

3. RESULTS AND DISCUSSION

We applied the procedure described above to members of Koronis and Veritas asteroid families. We selected a total of 96 asteroids of Koronis family in such a way to include in our sample for each condition exactly one quarter, that is 24 asteroids, that fulfill just that condition, and additional 24 asteroids that do not fulfill any of the conditions, when simplified Hamiltonian of Pavlović and Guzzo (2008) is used. Note that a vast majority of Koronis family members do in fact fulfill some condition; that they actually fulfill different conditions is of lesser importance, as fulfillment of any of them means that the generic condition of steepnes is fulfilled as well. Then we switched to the extended Hamiltonian and repeated the analysis, concentrating in particular to those asteroids which either ceased to fulfill any of the conditions, or began fulfilling one, although previously they did not fulfill any.

In Fig. 1 we show the fulfillment of the conditions for the analysed asteroids as computed by means of the simplified $h = h_0 + \varepsilon \mathcal{K}_0$ Hamiltonian (left), and of the extended $h = h_0 + \varepsilon \mathcal{K}_0 + \varepsilon \mathcal{K}_1$ Hamiltonian (right). Family members that changed

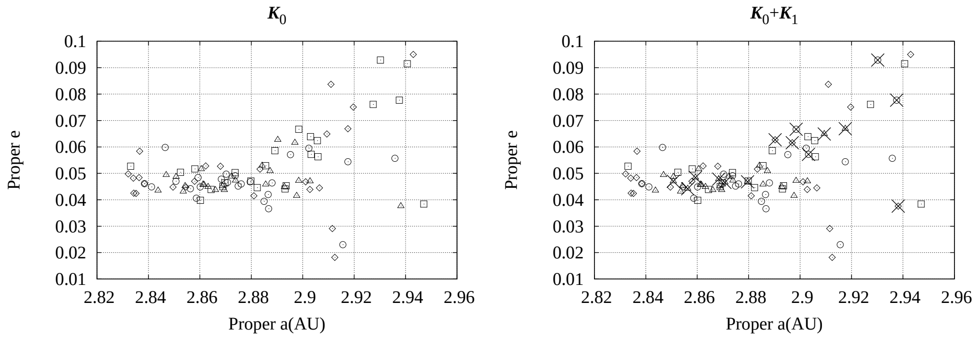

Figure 1: The fulfilled conditions for selected members of Koronis family as derived by using the simplified (left) and extended (right) Hamiltonian. Members fulfilling the convexity condition are marked by □, the quasi-convexity condition by ○, those fulfilling the 3-jet condition by △; asteroids for which none of the conditions are fulfilled are labeled ◊. Marked by × on the right are asteroids for which the fulfilled condition has changed.

the condition are marked by an × on the right hand side panel; they are 15 out of the total of 96 in our Koronis family sample. The most important change pertains, however, to the 3 asteroids which ceased to fulfill any condition (previously they all fulfilled the 3-jet one), while 2 bodies changed in the opposite sense. Thus the subsample of bodies not fulfilling any of the conditions increased by one when the more realistic Hamiltonian is used to probe the dynamics. We recall that for these bodies the Nekhoroshev theorem cannot be applied. Such a result could have actually been expected, as the selected asteroids are located far from the resonances included in the extended model, and their dynamics is not affected by the added terms.

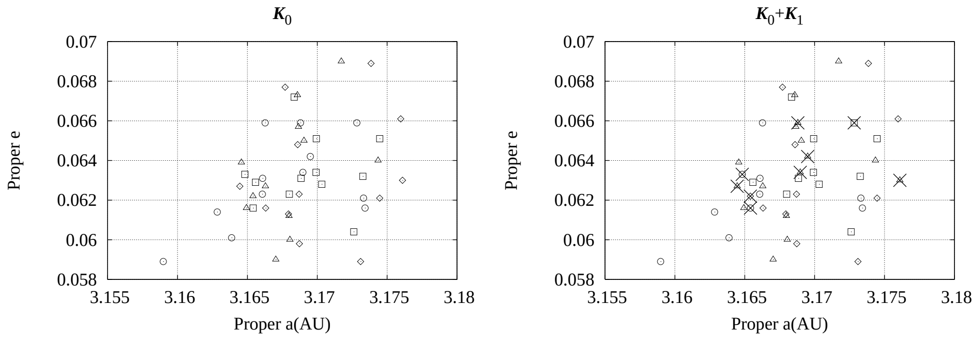

Figure 2: The same as Fig. 1, but for members of Veritas family.

A similar analysis for the Veritas family members yields also the similar results. Out of the 48 tested asteroids (12 per condition, and additional 12 not fullfilling any) only 9 have changed the condition they fulfilled previously; two bodies that did not fulfill any of the conditions now fulfill the 3-jet one, while one ceased to fulfill any condition. Thus, our subsample of bodies for which the theorem of Nekhoroshev cannot be applied has shrinked by one. Note that the above explanation involving the 'safe' location of the family holds in this case too. In Fig. 2 we show the changes of the fulfillment of the conditions for the Veritas family members.

Bearing in mind the above results, it was only natural to try to assess the fulfillment of the conditions for application of Nekhoroshev theorem for some resonant bodies. Hence, we performed tests for asteroids (582) Olympia and (945) Barcelona located near the strong secular resonance $\nu_5 = g - g_5$, where g and g_5 are the secular rates of the longitudes of perihelia of the asteroid and Jupiter, respectively. We found that for both objects the fulfillment of the conditions does not depend on the completeness of the Hamiltonian used. (582) Olympia fulfilled the strongest convexity condition in both tests, while (945) Barcelona did not fulfill any, again in both cases. This result might seem surprising, but, as shown by Morbidelli (1993), (582) is located very near, but outside the border of the resonance, while (945) appears to be inside. Anyway, these, as well as other cases of resonant dynamics require further and more detailed study, which we plan to conduct as part of our future work.

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