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DISCRETE SPACE THEORY OF RADIATIVE TRANSFER: APPLICATIONS

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Abstract. A concise description of the method of obtaining the solution of radiative transfer equation which can be applied to different geometrical and physical systems is given. This paper mainly deals with the steps for obtaining solution of transfer equation like (1) interaction principle for different geometries, (2) star product and (3) calculation of radiation field at internal points, and their applications in stellar atmosphere problems; in particular (a) study of reflection effect in close binary systems and (b) stellar winds in O and B type stars are discussed with their results.

1. INTRODUCTION

The study of transfer of radiation is pursued in several areas of scientific research such as astrophysics, reactor physics, meteorology and many other research fields. Astrophysics is one of the earliest areas of activity in which serious studies took place. It was George Stokes (1852, 1862) who first developed the concept of invariance in his glass plate problem. He showed how the reflectance remains invariant when an additional glass plate is added to a system of parallel plates of glass plates. Lord Rayleigh (1920) obtained transmission and reflection factors in more complicated situations such as dispersive medium.

The idea of addition of layer of arbitrary thickness to semi-infinite layer was proposed by Ambarzumian who noticed that the reflection characteristics remain invariant in such situations. This was taken up by Chandrasekhar who solved several problems through H-functions in a semi-infinite medium and X & Y functions in a finite medium. So layers with general properties can be added and their transmission and reflection properties can be calculated directly by utilizing the "Interaction Principle" (Redheffer 1962, Preisendorfer 1965, Grant and Hunt 1969a, 1969b), a generalized form of the invariance principle. Interaction principle is nothing but a manifestation of the conservation of radiant flux. It balances the emergent radiation with that of reflected and transmitted input radiation together with the internally generated radiation.

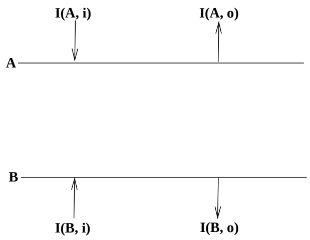

Figure 1: The interaction principle for one dimensional slab.

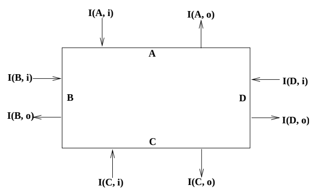

Figure 2: The interaction principle for two dimensional plane.

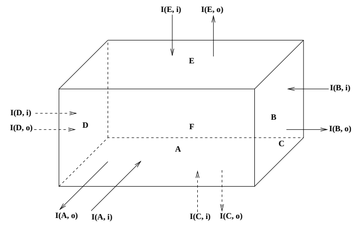

Figure 3: The interaction principle for three dimensional cuboid.

2. INTERACTION PRINCIPLE

Let I represents the specific intensity and i and o represent the incident and output intensities at the boundary of A, B in one-dimensional geometry, A B C D in two-dimensional geometry and at surfaces A B C D E F in three-dimensional geometry. Let t and r represent the transmission and reflection operators and $S(A), S(B)$ represent the internal source term.

2. 1. INTERACTION PRINCIPLE FOR: (A) ONE (B) TWO AND (C) THREE DIMENSIONAL GEOMETRY AND (D) SPHERICALLY SYMMETRIC ATMOSPHERES

(A) The output intensities in terms of the input intensities in Fig. 1 are as follows:

$$I(\mathrm{A}, o) = t(\mathrm{B},\mathrm{A})I(\mathrm{B}, i) + r(\mathrm{A},\mathrm{A})I(\mathrm{A}, i) + S(\mathrm{A}) \tag{1}$$

$$I(\mathrm{B}, o) = t(\mathrm{A},\mathrm{B})I(\mathrm{A}, i) + r(\mathrm{B},\mathrm{B})I(\mathrm{B}, i) + S(\mathrm{B})\ . \tag{2}$$

The above can be written in a matrix form

$$\begin{pmatrix} I(\mathrm{A}, o) \\ I(\mathrm{B}, o) \end{pmatrix} = \begin{pmatrix} r(\mathrm{A},\mathrm{A}) & t(\mathrm{B},\mathrm{A}) \\ t(\mathrm{A},\mathrm{B}) & r(\mathrm{B},\mathrm{B}) \end{pmatrix} \begin{pmatrix} I(\mathrm{A}, i) \\ I(\mathrm{B}, i) \end{pmatrix} + \begin{pmatrix} S(\mathrm{A}) \\ S(\mathrm{B}) \end{pmatrix}. \tag{3}$$

(B) The output intensities in terms of the input intensities in Fig. 2 are as follows:

$$I(\mathrm{A}, o) = r(\mathrm{A},\mathrm{A})I(\mathrm{A}, i) + t(\mathrm{B},\mathrm{A})I(\mathrm{B}, i) + t(\mathrm{C},\mathrm{A})I(\mathrm{C}, i) + t(\mathrm{D},\mathrm{A})I(\mathrm{D}, i) + S(\mathrm{A}) \tag{4}$$

Similarly one can write I(B, o), I(C, o), I(D, o), and get Eq (4) in matrix form (Peraiah 1984).

(C) For three dimensional geometry the output intensities in terms of the input intensities in Fig. 3 are as follows:

$$\begin{aligned} I(\mathrm{A}, o) &= t(\mathrm{A},\mathrm{B})I(\mathrm{B}, i) + t(\mathrm{A},\mathrm{C})I(\mathrm{C}, i) + t(\mathrm{A},\mathrm{D})I(\mathrm{D}, i) + t(\mathrm{A},\mathrm{E})I(\mathrm{E}, i) \\ &+ t(\mathrm{A},\mathrm{F})I(\mathrm{F}, i) + r(\mathrm{A},\mathrm{A})I(\mathrm{A}, i) + S(\mathrm{A}) \end{aligned} \tag{5}$$

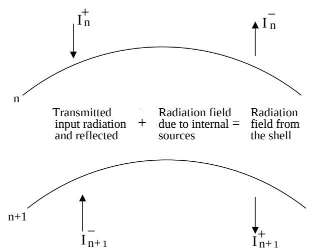

Figure 4: Schematic diagram of the interaction principle.

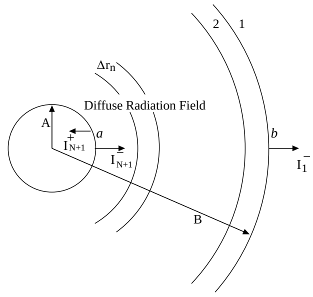

Figure 5: Schematic diagram of the diffuse radiation.

Similarly one can write I(B, o), I(C, o), I(D, o), I(E, o), I(F, o). This can also be written in the matrix form (Peraiah 1984) which is called 26-point problem (Preisendorfer 1965).

(D) For spherically symmetric atmospheres output intensities in terms of input intensities: If there is a shell with boundaries (n, n+1) shown in Fig. 4, the interaction principle can be written as follows:

$$\begin{pmatrix} \mathbf{I}_{n+1}^+ \\ \mathbf{I}_n^- \end{pmatrix} = \mathbf{S}(n, n+1) \begin{pmatrix} \mathbf{I}_n^+ \\ \mathbf{I}_{n+1}^- \end{pmatrix} + \sum(n, n+1) \ ; \tag{6}$$

see Peraiah (2002, Ch. 6, 155) for matrix $\mathbf{S}$(n, n+1) and $\sum$ (n, n+1).

2. 2. STAR PRODUCT

If there is another shell with boundaries (n+1, n+2) adjacent to (n, n+1), interaction principle for this shell can be written as

$$\begin{pmatrix} \mathbf{I}_{n+2}^+ \\ \mathbf{I}_{n+1}^- \end{pmatrix} = \mathbf{S}(n+1, n+2) \begin{pmatrix} \mathbf{I}_{n+1}^+ \\ \mathbf{I}_{n+2}^- \end{pmatrix} + \sum(n+1, n+2) \tag{7}$$

where $\mathbf{S}$(n+1, n+2) is also similarly defined. If we combine the two shells (n, n+1) and (n+1, n+2), then the interaction principle for the combined shell is written as

$$\begin{pmatrix} \mathbf{I}_{n+2}^+ \\ \mathbf{I}_n^- \end{pmatrix} = \mathbf{S}(n, n+2) \begin{pmatrix} \mathbf{I}_n^+ \\ \mathbf{I}_{n+2}^- \end{pmatrix} + \sum(n, n+2). \tag{8}$$

Redheffer (1962) calls $\mathbf{S}$(n, n+2) the star product of the two S-matrices $\mathbf{S}$(n, n+1) and $\mathbf{S}$(n+1, n+2) written as

$$\mathbf{S}(n, n+2) = \mathbf{S}(n, n+1) \star \mathbf{S}(n+1, n+2). \tag{9}$$

For other details refer to Redheffer (1962).

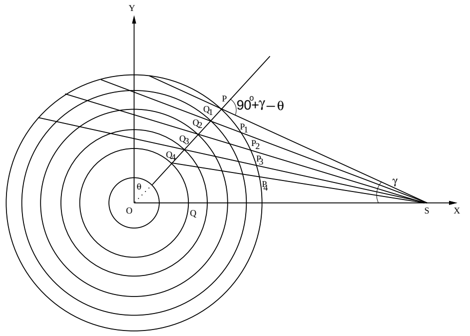

Figure 6: Calculation of radiation from a point source; Peraiah (1983).

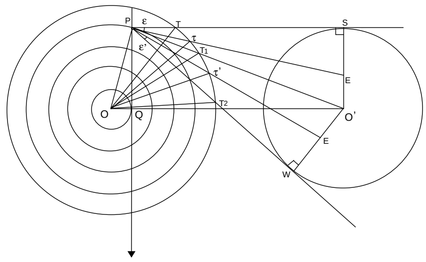

Figure 7: Calculation of radiation from an extended surface; Peraiah & Srinivasa Rao (1983).

3. METHOD OF OBTAINING THE SOLUTION OF TRANSFER EQUATION

We shall consider equation of transfer with monochromatic radiation field scattering isotropically and follow the steps to obtain the solution of transfer equation
(i) We divide the medium as shown in Fig. 5 into a number of "cells" whose thickness is less than or equal to the critical (τ_{crit}). The critical thickness is determined on the basis of physical characteristics of the medium. τ_{crit} ensures the stability and uniqueness of the solution.
(ii) Now the integration of the transfer equation is performed on the "cell" which is two-dimensional radius - angle grid bounded by $[r_n, r_{n+1}] \times [\mu_{j-\frac{1}{2}}, \mu_{j+\frac{1}{2}}]$ where $\mu_{j+\frac{1}{2}} = \sum_{k=1}^{j} C_k, j = 1, 2 \ldots, J$, where C_k are the weights of Gauss-Legendre formula.
(iii) By using the interaction principle described in Peraiah and Grant (1973), we obtain the reflection and transmission operators over the "cell".
(iv) Finally we combine all the cells by the star algorithm described in Peraiah and Grant (1973) and obtain the radiation field.
(v) Using discrete space theory the solution is developed for a non-LTE two level approximation case. We represent the integration by summation, derivatives by the differences for frequency (x), angle (μ) and radius (r). For obtaining solution the above mentioned steps (i)-(iv) has to be followed and for details refer to Peraiah (2002, Ch. 6, 179).

4. APPLICATIONS OF RADIATIVE TRANSFER EQUATION IN SPHERICAL SYMMETRY

The Discrete Space Theory (DST) method is a finite difference, discrete ordinate method which is a very accurate and easy to generalize to different physical problems. It is applied to 1) reflection effect in close binaries for calculating the self radiation of the primary component in a binary system, 2) stellar winds in O and B type stars: Hydrogen atmosphere 3) to study the effects of aberration and advection in a plane parallel medium in motion 4) K1 emission lines from circumstellar shells of N-

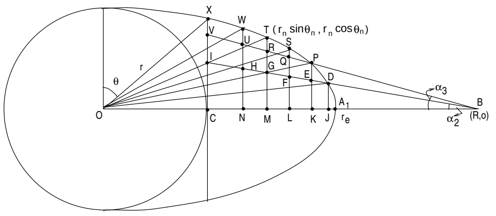

Figure 8: Schematic diagram showing distorted atmosphere of the component; Peraiah & Srinivasa Rao (2002).

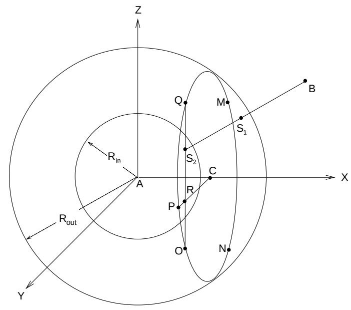

Figure 9: Schematic diagram of irradiation in Cartesian co-ordinate system; Srinivasa Rao & Varghese (2009).

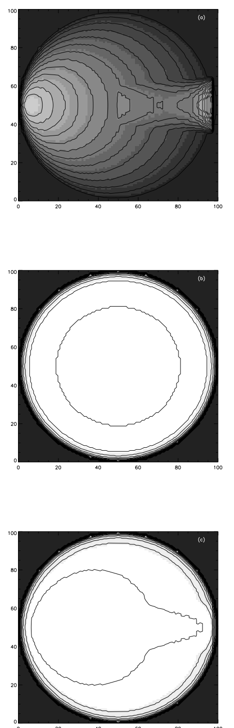

Figure 10: Contour plot for case1 for (a) reflected radiation (b) self radiation (c) total radiation; Srinivasa Rao & Varghese (2009).

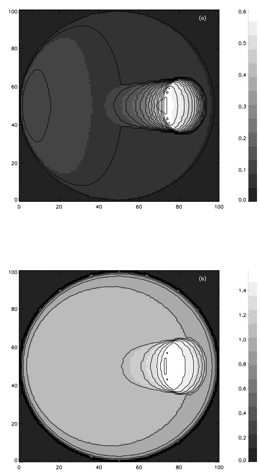

Figure 11: Contour map for (a) reflected and (b) total radiation for case 2; Srinivasa Rao & Varghese (2009).

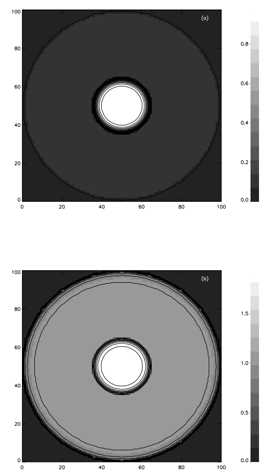

Figure 12: Contour map for (a) reflected and (b) total radiation for case 3; Srinivasa Rao & Varghese (2009).

Type star R Scl and 5) non-magnetic resonance scattering polarization in spherically symmetric media, polarized line formation in moving media, CRD, PRD and many more other applications. Here, the first two cases are presented.

5. RESULTS AND DISCUSSION

5. 1. REFLECTION EFFECTS IN CBs

Stars are assumed to be spherical with the atmosphere of the primary star divided into many shells as shown in Figs 6-7. The radiation comes from a distance outside the primary star. The main aim is to calculate radiation field on the primary component which is irradiated by the secondary component in a binary system. The calculation of radiation field is done by Rod model of one dimensional radiative transfer along the ray path inside the primary component. In addition to this radiation, we have the radiation of the star itself which is calculated by employing the radiative transfer equation in spherical symmetric approximation. In the next step the emergent radiation field is calculated by using formal solution of radiative transfer equation. The irradiation from the secondary component is given as a ratio $\frac{I_Q}{I_S}$, where I_Q is the intensity of radiation incident spherically symmetric inside the star and I_S the intensity coming from secondary component.

Incident radiation from a point source, Fig. 6:

- It is noticed that maximum radiation comes from intermediate points of the atmosphere, the reason being that intermediate layers have combined radiation from the star together with the incident radiation from a point source outside the star (for details see Peraiah 1983).

Incident radiation from an extended source, Fig. 7:

- When irradiation comes from an extended component, it is found that maximum radiation comes from outer layers of the primary component because part of the star is illuminated by the secondary component (see Peraiah & Srinivasa Rao 1983)

Effects of reflection on spectral lines:

- The effects of reflection on the line profiles are studied and the importance of reflection noticed. It is found that reflection fills the absorption profile at all frequency points, and also obtains P-cygni type profiles (for details see Peraiah & Srinivasa Rao 1998)

Distortion due to rotation and tidal forces, Fig. 8:

Transfer of line radiation in the atmospheres of close binary components whose atmosphere is distorted by self radiation and tidal forces due to the presence of the secondary component is studied. The distortion is measured in terms of the ratio of angular velocities at the equator and pole, mass ratio of the two components, the ratio of centrifugal force to that of gravity at the equator (f) and the ratio of the equatorial radius to the distance between the centers of gravity. The obtained equation of the distorted surface is a seventh degree equation which contains the above parameters.

- Peraiah & Srinivasa Rao (2002) noticed that the expansion of the medium produces P Cygni type profiles and that the irradiation enhances the emission in the lines, although the equivalent widths are reduced considerably. They have also suggested that in the modeling of the binary system it is necessary to include the effects of reflection.

Calculation of irradiation in 3D geometry, Fig. 9:

- Srinivasa Rao & Varghese (2009) calculated the reflected radiation from an extended surface of the primary component of a close binary system in 3-dimensional Cartesian coordinate geometry. They assumed that the secondary component is a point source moving in a circular orbit. The specific intensity of the radiation field of the primary component is estimated along the line of sight of an observer at infinity.

  They have set the coordinates of the secondary component as (x_2, y_2, z_2) where the radiation is incident on the primary component. The following cases were considered to calculate the direction cosines of the lines which are parallel to the Z-axis and also parallel to the line of sight. In the calculations secondary component was positioned at different places –

  Case 1: $x_2 = \mathrm{R}$, $y_2 = 0$, $z_2 = 0$; Case 2: $x_2 = \mathrm{R}\sin\frac{\pi}{4}$, $y_2 = 0$, $z_2 = \mathrm{R}\cos\frac{\pi}{4}$; and Case 3: $x_2 = 0$, $y_2 = 0$, $z_2 = \mathrm{R}$. It is noticed that the radiation field changes depending on the position of the secondary component as shown in the surface contours (Figs 10-12).

5. 2. STELLAR WINDS IN O AND B TYPE STARS: HYDROGEN ATMOSPHERE

In an isothermal atmosphere, Peraiah & Varghese (1993) solved the equations of conservation of mass and momentum along with line transfer equation. Radiation pressure is calculated in the line on the outward momentum of the outer layers of O and B type stars. Lines in Lyman, Balmer, Paschen and Brackett series are used to calculate the line radiation pressure assuming 10 levels of H atom. They employed Saha equation for estimating ionized H atoms, Boltzmann equation for calculating occupation numbers in different stages of excitation and non-LTE two level atom approximation of line transfer equation is considered for calculating the radiation pressure. They considered velocity law given by Blomme (1990) and showed the relation graphically (refer to Fig. 1 of Peraiah & Varghese 1993). It was noticed that hot stars show steep profile and cool stars have smaller acceleration. The primary idea is to derive terminal velocities of the winds in O and B stars. Using above procedure they obtained required parameters for calculating line profiles along the line of sight and compared them with the observations.

6. MERITS AND DEMERITS OF THE METHOD:

- The DST method can be generalized for more complicated physical situations like 2D and 3D-geometry

- Accuracy and stability of the numerical solution can be checked up to the machine accuracy

- Self consistency checks like reflection and transmission matrices, flux conservation is used to ensure the correct coding and accuracy of the discrete representation of the method.
- Easy parallelization and vectorization of the code
- The algorithm is written in such a way that the code can be written for parametric study. While changing the required parameters one can compare the results with different geometries of the problem like plane parallel to spherical and comoving frame to rest frame etc.

⋆ Application to the realistic atmospheric modeling leads to increased dimensionality of the matrices in the algorithm and large memory and CPU time requirements.

7. CONCLUSION

The Discrete Space theory method, a finite difference, discrete ordinate method, is presented. It is very accurate method and can be generalized to different physical problems. At present only two applications of the method are discussed: 1) reflection effect in close binaries, 2) stellar winds in O and B type stars: hydrogen atmosphere. Many more applications are left out because of the limited size of the article.

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