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LOBACHEVSKY'S GEOMETRY AND RESEARCH OF GEOMETRY OF THE UNIVERSE

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Abstract. For the first time N. I. Lobachevsky gave a talk on the new geometry in 1826; three years after he had published a work "On the fundamentals of geometry", containing all fundamental theorems and methods of non-Euclidean geometry. A small part of the article was devoted to the study of geometry of the Universe. The interpretation of geometrical concepts in pure empirical way was typical for mathematicians at the beginning of the XIX century; in this connection it was important for scientists to find application of his geometry. Having the purpose to determine experimentally the properties of real physical Space, Lobachevsky decided to calculate the sum of angles in a huge triangle with two vertexes in opposite points of the terrestrial orbit and the third – on the remote star. Investigating the possibilities of solution of the set task, Lobachevsky faced the difficulties of theoretical, technical and methodological character. More detailed research of different aspects of the problem led Lobachevsky to the comprehension of impossibility to obtain the values required for the goal achievement, and he called his geometry an imaginary geometry.

The purely empirical interpretation of geometrical concepts was typical for mathematics of the early XIX century; therefore it was so important for N. Lobachevsky, the founder of non-Euclidean geometry, to find an example of realization of the new geometry in real physical space. The very nature could serve to be an ideal model of Lobachevsky's geometry if it were possible to prove that the cosmic space was non-Euclidean. In one of his letters he wrote: "I am getting more and more convinced that necessity of our geometry can not be proved, at least by a human mind . . . geometry has to be placed in one and the same rank not with arithmetic existing purely a priori but rather with mechanics".

Lobachevsky's reference to astronomical objects was not casual. He took a keen interest in astronomy during his student years at Kazan University. The founder of Astronomy Department of Kazan University (1810), a German scientist I.A. Littrov deemed N.I. Lobachevsky one of his best students. The mathematician made his first astronomical observations under the guidance of Littrov. At the beginning of his teaching career at Kazan University Lobachevsky taught not only mathematics, but also astronomy and physics. Becoming the rector, he directed the construction

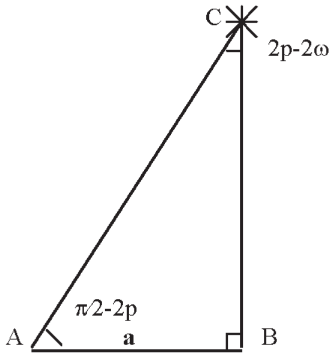

Figure 1: See the text.

of new astronomical observatory that became one of the best-equipped observatories in Russia.

For the first time N.I. Lobachevsky gave a talk on the new geometry in 1826; three years after he had published a work "On the fundamentals of geometry" (Лобачевский 1929) containing all fundamental theorems and methods of non-Euclidean geometry. Small part of this article (only two and half pages) was devoted to the study of geometry of the Universe (Лобачевский 1946a).

Let us examine this fragment devoted to "big" triangles. Having the purpose to determine experimentally the properties of real physical Space, Lobachevsky decided to calculate the sum of angles in a huge right triangle with two vertexes A and B in the opposite points of the terrestrial orbit and the third C - on the remote star.

In Figure 1 we can see Lobachevsky's space triangle: **a** is length of the terrestrial orbit diameter, $\mathbf{2}\omega$ is an angular sum defect of the triangle ABC, **2p** is the biggest parallax of a fixed star, $\angle B = \frac{\pi}{2}$, $\angle A = \frac{\pi}{2} - 2p$, $\angle C = 2p - 2\omega$.

In this case we can use right triangle, because it is possible from the astronomical standpoint and it is desirable for reasons of opportunity of the use of Lobachebsky's geometrical method to solve this problem. Investigating the possibilities of solution of the set task, Lobachevsky faced some difficulties of theoretical, technical and methodological character. The calculations of stellar parallaxes had always been based on Euclidean geometry. To avoid the problems connected with the use of these results, Lobachevsky developed a special technique of reasoning. Moreover it was impossible to calculate precisely value of the sum of angles in a huge right triangle or this sum defect because similarity of triangles does not exist in Lobachevskian geometry and trigonometric formulas for the solution of a triangle are not very useful, as we do not know the length accepted for a unit in this theory (radius of curvature of the space). In this case we can speak only about an estimate of the angular sum defect.

First of all Lobachevsky tried to find a lower bound for the parallax. As the fixed star is very distant from the Earth, we can suppose that in the triangle ABC the side

AC is almost parallel to BC. In this case it is necessary

$$\Pi(a) > \pi/2 - 2p,$$

Π(a) is the value of Lobschevsky's function.

As a result of the transformations Lobachevsky obtained such inequality

$$a < tg\ 2p,$$

in modern Lobachevskian geometry

$$\frac{a}{k} < tg\ 2p$$

$$2p > arctg\frac{a}{k},$$

where k is the radius of curvature of the space.

According to the result of Lobachevsky the parallax of an arbitrary distant star remains bigger than some constant depending on the radius of curvature. Lobachevsky did not have a special designation for this value. This inequality evoked objections of attentive readers. First, there is no real possibility to determine this value experimentally. Secondly the assertion that parallaxes of infinitely distant stars must remain larger that a certain constant, if $\frac{a}{k}$ is a constant, seemed strange. The question of the Universe size arose. Lobachevsky did not formulate the obtained result in an explicit form and did not give any comments thereto, possibly trying to avoid additional discussions.

Then Lobachevsky investigated the angular sum defect of the triangle ABC. He used more distant star parallax p' to characterize it. He found such an angular sum defect estimate:

$$\omega < 2p\sin^2\left(\frac{x}{2}\right), \quad \sin x = \frac{\sin p^I}{\sin p}\sqrt{\frac{\cos 2p}{\cos 2p^I}}.$$

In his reasoning Lobachevsky used the values of parallaxes derived by the French seaman and amateur astronomer, earl Dassa-Montdardier. His method of determination of stellar parallaxes was rather primitive and did not yield acceptable results. However, it should be noted, that at the beginning of the XIX century the attempts to measure parallaxes could not be called satisfactory; the obtained values were frequently far overrated. Nevertheless, the principles of choice of a parallax for the solution of the problem raise some questions.

Lobachevsky based his reasoning on Dassas-Montdardier's parallax values, namely, the least of the derived values – $p'= 0''.62$ for Sirius and the greatest $p = 1''$ for 29$^{\text{th}}$ Eridanus ("Keid" by Lobachevsky's nomenclature). He showed that in a triangle with the apex on Keid the angular sum defect 2ω proved to be less than $0''.43$. In Idelson's work (Идельсон 1975) it was noted that if we specify the values of parallaxes with regard to the modern data and accept $p' = 0''.05$ as the least parallax, and $= 0''.75$ as the greatest parallax, then we derive

$$2\omega < 0\overset{\prime\prime}{.}0033\ .$$

It means that we may assert that the upper bound of the angular sum defect is at least two orders less. So the less parallax of a distant star, the lower is the angular sum defect value in the smaller triangle.

If in a right triangle there are legs of lengths a and b, and angular sum is $\pi - 2\omega$, then:

$$tg\ \omega = \left(\frac{e^a - 1}{e^a + 1}\right) \cdot \left(\frac{e^b - 1}{e^b + 1}\right).$$

This means that the smaller a triangle is the less the angular sum defect ω will be. Lobachevsky noted that these reasonings demonstrate accuracy of Euclidean geometry and allow to consider the foundations of this geometry quasi proved (Лобачевский 1946a, p. 209).

He found estimation of the angular sum defect of a isosceles right triangle with legs equal to the diameter of the Earth orbit a= 3 × 10^8 km.

$$\omega < (p')^2 \times \sin 1''$$

$$2\omega < 0\overset{\prime\prime}{.}000372.$$

Professor of Kazan University A.P. Kotelnikov, author of the comments for Lobachevsky's *Opera Omnia* showed that Lobachevsky made a mistake or a misprint in the this inequality by increasing the estimate by 2 orders. In fact it must be

$$2\omega < 0\overset{\prime\prime}{.}00000372.$$

But even having Lobachevsky's value, the angular sum defect in the triangle within the limits of the Solar System proved to be too insignificant if we speak about realization of the non-Euclidean geometry. Lobachevsky failed to obtain the lower bound of the angular sum defect for such triangle. Thus, the said research did not yield any appreciable result.

N.I. Lobachevsky mentioned the astronomy-related fragment of the work "On the fundamentals of geometry" only once. In 1837 the translation of his essay "Imaginary geometry" into French (Lobatschewsky 1837) was published. This version of the work was written before the Russian version of "Imaginary geometry" (Лобачевский 1946b, p. 171) (though published later), it differed from the Franch one. It had a short reference to the fragment being discussed here: "In a different place, being guided by some astronomical observations, I proved that the angular sum in a triangle the sides of which have approximately the same value as the distance from the Earth to the Sun, will never differ from two right angles to the amount exceeding $0\overset{\prime\prime}{.}0003$. Besides, this value must be the less, the less the triangle sides are" (Лобачевский 1946b, p. 303). Here the author repeated the inaccurate estimation of the angular sum defect in a triangle with legs equal to the terrestrial orbit diameter (based on Dassas-Montdardier's parallaxes). This mentioning was not accompanied by any calculations or comments; further this reference was excluded from the later Russian version of "Imaginary geometry" published in the "Proceedings of Kazan University".

It is customary to think that solving the problem of Lobachevsky's geometry realization was impossible because the sum of triangle angles differed from π by a too little value, less than the values of an inaccuracy of measurement. N.I. Lobachevsky did not make attempts to specify the results obtained by him, using more precise values of stellar parallaxes.

At this time the problem of assessment of parallaxes was handled by V.Ya. Struve. In 1835 - 1838 he investigated the α-Lira parallax that he determined to be equal to $0\overset{''}{.}125$ + 0,055$''$. The preliminary results were published by him in 1837, in the book "Micrometric measurements" (Struve 1837, p. CLXII-CLXXIII), a small brochure about binary stars. Struve characterized his result as follows: "This result is very important as it shows that the parallax may not exceed a minor share of second and that the results obtained by Pazzi, Caladrelli and Brinkley who determined the α-Lira parallax to amount to several seconds are incorrect. On the other hand, my observations gave quite a definite value for this parallax - though minor, but much exceeding the error according to the theory of probabilities" (Струве 1953, p. 185). Quite precise results were obtained by F. Bessel for 61-Swan, by T. Henderson for α -Centauri, by F. Peters for 8 bright stars.

This scientific research could not escape Lobachevsky's attention. Why did Lobachevsky select the observational results of Dassa-Montdardier out of all possible parallax values? As indicated above Struve obtained too small parallax values which would inevitably lead to less values of the angular sum defect. There was no sense to make more exact the earlier obtained results. The problem consisted not so much in the fact that the angular sum defect value in a triangle proved to be less than the possible observation error, as in the fact that, having further improvement of measurement techniques and specification of parallax values, the upper bounds of the angular sum defect obtained by Lobachevsky, being of truly small value in their nature, could only decrease. This can be seen from the estimation of values provided above. More detailed research of different aspects of the problem led Lobachevsky to the comprehension of impossibility to obtain the values angular sum defect required for the goal achievement, and he called his geometry an imaginary geometry.

Finally Lobachevsky came to the following conclusion: "Well, not emphasizing that the space may be extended boundlessly in one's imagination, the Nature shows us such distances in comparison with that even the distances from our planet till motionless stars disappear because of their insignificance. One can not assert any more after this that the assumption stating that the measure of lines does not depend on the angles - assumption considered by many geometricians to be an indisputable law, not requiring any proofs – has appeared to be very probably false even before we trespass the limits of the world visible to us.

On the other hand, we are not able to comprehend the kind of relation existing in the natural environment, bringing together totally different values like lines and angles. **So, it is very much possible that the Euclidean views are the only true assertions, though they will remain unproved forever.**

Anyway, **the new Geometry**, the basis of which has already been laid, **even if it does not exist in the nature, still may exist in our imagination**, and, being not used for measurement, in fact it opens a new, extensive field for reciprocal use of Geometry and Analytics" (Лобачевский 1946a, p. 209-210).

It was after this work that the new geometry got the name of "imaginary geometry". The next essay of Lobachevsky (1835) was entitled "Imaginary geometry" (Лобачевский 1946b, p. 16-70). The astronomical research was not able to confirm the realization of Lobachevsky's geometry in physical space; the new geometry shifted from the area of reality to the field of imaginary. In his essay "Application of imaginary geometry to certain integrals" (Лобачевский 1946b, p. 181-294) Lobachevsky aimed to demonstrate the potential of application of the geometrical methods developed by him for solution of mathematical analysis problems, which in itself substantiated the right of the new geometry to exist.

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