APPLICABILITY OF NEKHOROSHEV’S THEOREM
IN SOME SELECTED CHAOTIC REGIONS

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Abstract. The Hamiltonian of an asteroid perturbed by major planets is analysed from
the point of view of the fulfillment of conditions for applying Nekhoroshev’s theorem, such
as the quasi-convexity or the 3-jet one, in the selected phase-space regions already known to
harbour chaotic motion.

1. INTRODUCTION
Several papers have been published so far that deal with the application of Nekhoro-
shev theorem (Nekhoroshev, 1977) to various, often simplified dynamical systems. So,
for example, Celleti and Giorgilli (Celletti and Giorgilli, 1991) directly apply the the-
orem in the framework of the restricted, circular three body problem in Lagrangian
equilibrium points, Celleti and Ferrara (Celletti and Ferrara, 1996) apply it to the
real Sun-Jupiter-Ceres system, considering again only the simplified dynamics, while
Guzzo et al. (Guzzo et al., 2002) employ it to study real N-body system, but using its
special formulation. In all these cases authors assume the fulfillment of conditions for
the application of the theorem. Only in the paper by Benettin et al. (1998) authors
give a detailed analysis of Hamiltonian of the restricted, circular problem of three
bodies in the Lagrangian points $L_4, L_5$, from the point of view of the fulfillment of
the condition of convexity, or alternatively of less strict quasi-convexity, “directional
quasi-convexity” or of rather loose 3-jet condition.

In the present paper we present an analysis of fulfillment of the quasi-convexity
condition for the Hamiltonian of a system consisting of an asteroid perturbed by
Jupiter and Saturn.

2. THEORETICAL FRAMEWORK
We start from Hamiltonian in heliocentric coordinates:

$$\mathcal{H} = \mathcal{H}_0(L) + \varepsilon \mathcal{H}_1(L, G, H, l, g, h, a', e', i', l', \omega', \Omega')$$

(1)
where

\[\begin{align*}
L &= \sqrt{\alpha} = p, \\
G &= \sqrt{\alpha} \sqrt{1 - e^2} = p, \\
H &= \sqrt{\alpha} \sqrt{1 - e^2 \cos i} = \omega
\end{align*}\]

are the usual Delaunay’s variables.

Primed quantities refer to the perturbing planet with \( \varepsilon = m' \) denoting its mass. \( \mathcal{H}_1 \) is given as sum over all perturbing planets.

The short periodic terms are eliminated by means of the Lie algorithm, that is averaging over the mean anomalies (see Milani and Knežević, 1999), so that the resulting secular Hamiltonian (1) depends only on mean elements:

\[
\tilde{\mathcal{H}} = \mathcal{H}_0(L) + \varepsilon \mathcal{H}_1(L, \bar{G}, \bar{H}, -\bar{g}, \bar{h}, a', e', -\omega', \Omega') + \\
+ \varepsilon^2 \mathcal{H}_2(L, \bar{G}, \bar{H}, -\bar{g}, \bar{h}, a', e', -\omega', \Omega') + O(\varepsilon^3)
\]

\( \mathcal{H}_0 \) is two body Hamiltonian, while action \( L \) is a constant, and thus a first quasi-invariant of motion, that is a proper element. Let us denote \( \mathcal{K} \) a truncated Hamiltonian, i.e. \( \mathcal{K} = \mathcal{H}_1 + \varepsilon^2 \mathcal{H}_2 \). Hamiltonian \( \mathcal{K} \) can be written as a sum of unperturbed part \( \mathcal{K}_0 \) and of perturbation \( \mathcal{K}_P \). This particular form has been introduced by Williams (1969), who expanded \( \mathcal{K} \) in Taylor series in small eccentricities and inclination of the planets (see Lamaitre and Morbidelli, 1994):

\[
\mathcal{K} = \mathcal{K}_0 + \mathcal{K}_1 + \mathcal{K}_2 + \cdots
\]

Kozai (1962) demonstrated that zero-order term \( \mathcal{K}_0 \) (commonly known as Kozai’s Hamiltonian) depends only on one angle (argument of perihelion \( g \)), and that it is thus integrable.

Introducing suitable action-angle variables, Kozai’s Hamiltonian can be expressed as a function of actions only:

\[
\mathcal{K} = \mathcal{K}_0(J, Z) + \mathcal{K}_1(\psi, z, J, Z).
\]

\( J, Z \) are actions, \( \psi, z \) are conjugated angles and the perturbation contains only terms linear in \( e', i' \). Canonical transformation from Delaunay’s variables (2) to the action-angle ones cannot be explicitly expressed, but can be performed numerically. The dynamics described by Kozai’s Hamiltonian is characterized by a couple of frequencies:

\[
\omega_\psi = \frac{\partial \mathcal{K}}{\partial J}, \quad \omega_z = \frac{\partial \mathcal{K}}{\partial Z}.
\]

Computation of the perturbation \( \mathcal{K}_1 \) is performed with respect to a reference orbit of the \( \mathcal{K}_0 \) dynamics.
Applying Henrard’s seminumerical method (Henrard, 1990) on Hamiltonian (5) one eliminates the long periodic angles $\psi$ and $z$ (see Lemaître and Morbidelli, 1994 for details of the procedure) and gets:

$$\tilde{\mathcal{K}} = \tilde{K}_0(\tilde{J}, \tilde{Z}) + \tilde{K}_1(\tilde{J}, \tilde{Z}) + \mathcal{R}(\tilde{J}, \tilde{Z}, \tilde{\psi}, \tilde{z})$$  \hspace{1cm} (7)

where barred quantities are new action-angle variables, called “proper elements” of the asteroid. Function $\mathcal{R}$ is a remainder and contains higher degree terms. Using described transformations, Hamiltonian (3) can be written as a function of asteroid proper elements, and only now Hamiltonian can be tested against fulfillment of conditions for application of the theorem of Nekhoroshev. Setting:

$$\tilde{\mathcal{H}}_s = \mathcal{H}_0(L) + \varepsilon(K_0(L, J, Z) + K_1(L, J, Z)),$$  \hspace{1cm} (8)

and introducing for simplicity $(L, J, Z) = (I_1, I_2, I_3)$, the condition of quasi-convexity of the integrable Hamiltonian $\mathcal{H}_s$ after Nekhoroshev (1977) is that the system of equations:

$$\sum_{i=1}^{3} \frac{\partial \mathcal{H}_s}{\partial I_i} \eta_i = 0,$$  \hspace{1cm} (9)

$$\sum_{i,j=1}^{3} \frac{\partial^2 \mathcal{H}_s}{\partial I_i \partial I_j} \eta_i \eta_j = 0$$  \hspace{1cm} (10)

does not have real solutions except trivial one $\eta = 0$. More formally: Hamiltonian $\mathcal{H}_s$ is quasi-convex in $I$ when restriction of Hessian on hypersurface which is orthogonal on the frequency vector $\omega$ is positively or negatively definite.

In practice, one first computes Hessian matrix $A_{i,j} = \frac{\partial^2 \mathcal{H}_s}{\partial I_i \partial I_j}$ and vector $\omega = \left(\frac{\partial \mathcal{H}_s}{\partial I_1}, \frac{\partial \mathcal{H}_s}{\partial I_2}, \frac{\partial \mathcal{H}_s}{\partial I_3}\right)$ defining an orthogonal hypersurface $\Pi(\omega)$. Then, coordinates $I$ are rotated until vector $\omega$ does not coincide with first coordinate axis of matrix $A$. Remaining $2 \times 2$ block defines matrix $B$. When both eigen values of $B$ have the same sign, Hamiltonian $\mathcal{H}_s$ is quasi-convex.

3-jet condition is more complex, because it requires derivatives of the Hamilton function up to third order. Consequently, in addition to (9) and (10) we have:

$$\sum_{i,j,k=1}^{3} \frac{\partial^3 \mathcal{H}_s}{\partial I_i \partial I_j \partial I_k} \eta_i \eta_j \eta_k = 0$$  \hspace{1cm} (11)

Practical verification of these condition requires first to find critical points $(u_-, u_+)$ of system (10), and then to check condition (11). Results of these check are given in the last two columns of Table 1.
3. RESULTS

As aforementioned, the present paper concentrates on the analysis of the fulfillment of quasi-convexity and 3-jet conditions of the Hamiltonian for several selected members of Veritas asteroid family, located in the outer part of the main belt with an average proper semimajor axis of about 3.174 AU. This family has already been studied by applying the spectral formulation of the theorem of Nekhoroshev (1977). The spectral formulation is based on the Fourier analysis of a suitably chosen function of time series of orbital elements obtained by means of the numerical integration covering up to 100 Myr (Guzzo et al., 2002). An example of such a spectrum is shown in Fig. 1 and exhibits a clear continuous structure indicating that the object is not in the Nekhoroshev regime.

The results of the present analysis are given in Table 1. They indicate that Hamilton function for the considered members of Veritas family is NOT quasi-convex, because the eigen values $\lambda_1$ and $\lambda_2$ of matrix $B$ are of opposite sign. From two last columns of Table 1 one can easily see that values of $K_1$ corresponding to critical points $u_-$ and $u_+$ are small (of the order of $\varepsilon^* < 10^{-5}$) for all selected asteroids, except for 2428 Kamenyar, where we get an order of magnitude larger values. Hence, we can conclude that for all asteroids, except for Kamenyar, 3-jet condition is not fulfilled.
Table 1: Calculated Eigen values of $B$ and values of $\mathcal{K}_1$.

<table>
<thead>
<tr>
<th>Number</th>
<th>Asteroid</th>
<th>Eigen values of $B$</th>
<th>$\mathcal{K}_1$</th>
<th>$(u_-)$</th>
<th>$(u_+)$</th>
</tr>
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<tr>
<td>490</td>
<td>Veritas</td>
<td>-0.13776360D-02</td>
<td>0.67849412D+01</td>
<td>0.2248D-05</td>
<td>0.1518D-05</td>
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<td>2147</td>
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<td>0.29421851D+01</td>
<td>0.9742D-05</td>
<td>0.9511D-05</td>
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<td>2428</td>
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<td>0.78520296D+01</td>
<td>0.4225D-04</td>
<td>0.9454D-04</td>
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<td>3542</td>
<td>Tanjiazhen</td>
<td>-0.39531112D+00</td>
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<td>0.9377D-05</td>
</tr>
</tbody>
</table>

Thus we can say that we have, for a sample of selected Veritas family members, confirmed results by Guzzo et al. (2002) in a completely independent way.

References