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APPLICABILITY OF NEKHOROSHEV'S THEOREM IN SOME SELECTED CHAOTIC REGIONS

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Abstract. The Hamiltonian of an asteroid perturbed by major planets is analysed from the point of view of the fulfilment of conditions for applying Nekhoroshev's theorem, such as the quasi-convexity or the 3-jet one, in the selected phase-space regions already known to harbour chaotic motion.

1. INTRODUCTION

Several papers have been published so far that deal with the application of Nekhoroshev theorem (Nekhoroshev, 1977) to various, often simplified dynamical systems. So, for example, Celleti and Giorgilli (Celletti and Giorgilli, 1991) directly apply the theorem in the framework of the restricted, circular three body problem in Lagrangian equilibrium points, Celleti and Ferrara (Celletti and Ferrara, 1996) apply it to the real Sun-Jupiter-Ceres system, considering again only the simplified dynamics, while Guzzo et al. (Guzzo et al., 2002) employ it to study real N-body system, but using its special formulation. In all these cases authors assume the fulfillment of conditions for the application of the theorem. Only in the paper by Benettin et al. (1998) authors give a detailed analysis of Hamiltonian of the restricted, circular problem of three bodies in the Lagrangian points L_4, L_5, from the point of view of the fulfillment of the condition of convexity, or alternatively of less strict quasi-convexity, "directional quasi-convexity" or of rather loose 3-jet condition.

In the present paper we present an analysis of fulfillment of the quasi-convexity condition for the Hamiltonian of a system consisting of an asteroid perturbed by Jupiter and Saturn.

2. THEORETICAL FRAMEWORK

We start from Hamiltonian in heliocentric coordinates:

$$\mathcal{H} = \mathcal{H}_0(L) + \varepsilon\mathcal{H}_1(L, G, H, l, g, h, a', e', i', l', \omega', \Omega') \tag{1}$$

where

$$
\begin{array}{rclcrcl}
L & = & \sqrt{a} & , & l & = & M \\
G & = & \sqrt{a}\sqrt{1-e^2} & , & g & = & \omega \\
H & = & \sqrt{a}\sqrt{1-e^2}\cos i & , & h & = & \Omega,
\end{array}
\tag{2}
$$

are the usual Delaunay's variables.

Primed quantities refer to the perturbing planet with $\varepsilon = m'$ denoting its mass. $\mathcal{H}_1$ is given as sum over all perturbing planets.

The short periodic terms are eliminated by means of the Lie algorithm, that is averaging over the mean anomalies (see Milani and Knežević, 1999), so that the resulting secular Hamiltonian (1) depends only on mean elements:

$$
\begin{aligned}
\bar{\mathcal{H}} = \bar{\mathcal{H}}_0(\bar{L}) + \varepsilon\bar{\mathcal{H}}_1(\bar{L}, \bar{G}, \bar{H}, -, \bar{g}, \bar{h}, a', e', i', -, \omega', \Omega') + \\
+\varepsilon^2\bar{\mathcal{H}}_2(\bar{L}, \bar{G}, \bar{H}, -, \bar{g}, \bar{h}, a', e', i', -, \omega', \Omega') + O(\varepsilon^3)
\end{aligned}
\tag{3}
$$

$\bar{\mathcal{H}}_0$ is two body Hamiltonian, while action $\bar{L}$ is a constant, and thus a first quasi-invariant of motion, that is a proper element. Let us denote $\mathcal{K}$ a truncated Hamiltonian, i.e. $\mathcal{K} = \varepsilon\mathcal{H}_1 + \varepsilon^2\mathcal{H}_2$. Hamiltonian $\mathcal{K}$ can be written as a sum of unperturbed part $\mathcal{K}_0$ and of perturbation $\mathcal{K}_P$. This particular form has been introduced by Williams (1969), who expanded $\mathcal{K}$ in Taylor series in small eccentricities and inclination of the planets (see Lamaitre and Morbidelli, 1994):

$$
\mathcal{K} = \mathcal{K}_0 + \mathcal{K}_1 + \mathcal{K}_2 + \cdots \tag{4}
$$

Kozai (1962) demonstrated that zero-order term $\mathcal{K}_0$ (commonly known as Kozai's Hamiltonian) depends only on one angle (argument of perihelion g), and that it is thus integrable.

Introducing suitable action-angle variables, Kozai's Hamiltonian can be expressed as a function of actions only :

$$
\mathcal{K} = \mathcal{K}_0(J, Z) + \mathcal{K}_1(\psi, z, J, Z). \tag{5}
$$

J, Z are actions, ψ, z are conjugated angles and the perturbation contains only terms linear in e', i'. Canonical transformation from Delaunay's variables (2) to the action-angle ones cannot be explicitly expressed, but can be performed numerically. The dynamics described by Kozai's Hamiltonian is characterized by a couple of frequencies:

$$
\omega_\psi = \frac{\partial\mathcal{K}}{\partial J}\ , \quad \omega_z = \frac{\partial\mathcal{K}}{\partial Z}\ . \tag{6}
$$

Computation of the perturbation $\mathcal{K}_1$ is performed with respect to a reference orbit of the $\mathcal{K}_0$ dynamics.

Applying Henrard's seminumerical method (Henrard, 1990) on Hamiltonian (5) one eliminates the long periodic angles ψ and z (see Lemmaitre and Morbidelli, 1994 for details of the procedure) and gets:

$$\bar{\mathcal{K}} = \bar{\mathcal{K}}_0(\bar{J}, \bar{Z}) + \bar{\mathcal{K}}_1(\bar{J}, \bar{Z}) + \mathcal{R}(\bar{J}, \bar{Z}, \bar{\psi}, \bar{z}) \tag{7}$$

where barred quantities are new action-angle variables, called "proper elements" of the asteroid. Function $\mathcal{R}$ is a remainder and contains higher degree terms.
Using described transfrmations, Hamiltonian (3) can be written as a function of asteroid proper elements, and only now Hamiltonian can be tested against fulfillment of conditions for application of the theorem of Nekhoroshev. Setting:

$$\bar{\mathcal{H}}_s = \mathcal{H}_0(L) + \varepsilon(\mathcal{K}_0(L, J, Z) + \mathcal{K}_1(L, J, Z)), \tag{8}$$

and introducing for simplicity $(L, J, Z) = (I_1, I_2, I_3)$, the condition of quasi-convexity of the integrable Hamiltonian $\mathcal{H}_s$ after Nekhoroshev (1977) is that the system of equations:

$$\sum_{i=1}^{3} \frac{\partial \mathcal{H}_s}{\partial I_i} \eta_i = 0, \tag{9}$$

$$\sum_{i,j=1}^{3} \frac{\partial^2 \mathcal{H}_s}{\partial I_i \partial I_j} \eta_i \eta_j = 0 \tag{10}$$

does not have real solutions except trivial one $\eta = 0$. More formally: Hamiltonian $\mathcal{H}_s$ is quasi-convex in I when restriction of Hessian on hypersurface which is orthogonal on the frequency vector ω is positively or negatively definite.

In practice, one first computes Hessian matrix $A_{i,j} = \frac{\partial^2 \mathcal{H}_s}{\partial I_i \partial I_j}$ and vector $\omega = \left(\frac{\partial \mathcal{H}_s}{\partial I_1}, \frac{\partial \mathcal{H}_s}{\partial I_2}, \frac{\partial \mathcal{H}_s}{\partial I_3}\right)$ defining an ortoghonal hypersurface $\Pi(\omega)$. Then, coordinates I are rotated until vector ω does not coincide with first coordinate axis of matrix A. Remaining 2×2 block defines matrix B. When both eigen values of B have the same sign, Hamiltonian $\mathcal{H}_s$ is quasi-convex.

3-jet condition is more complex, because it requires derivatives of the Hamilton function up to third order. Consequently, in addition to (9) and (10) we have:

$$\sum_{i,j,k=1}^{3} \frac{\partial^3 \mathcal{H}_s}{\partial I_i \partial I_j \partial I_k} \eta_i \eta_j \eta_k = 0 \tag{11}$$

Practical verification of these condition requires first to find critical points (u_-,u_+) of system (10), and then to check condition (11). Results of these check are given in the last two columns of Table 1.

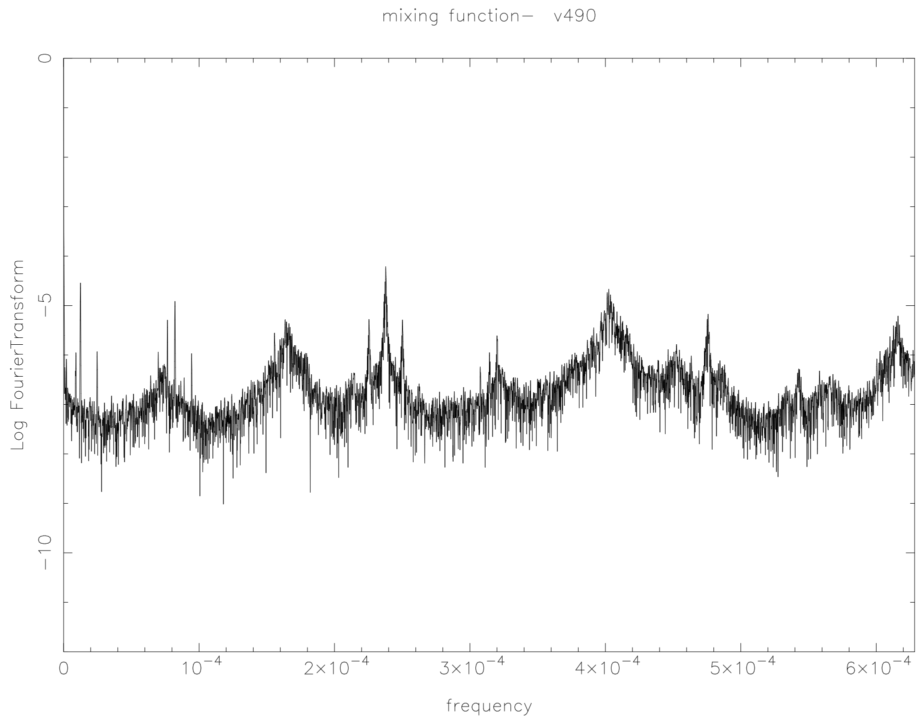

Figure 1: Continuous Fourier spectrum (Guzzo et al., 2002) for asteroid 490 Veritas.

3. RESULTS

As aforementioned, the present paper concentrates on the analysis of the fulfillment of quasi-convexity and 3–jet conditions of the Hamiltonian for several selected members of Veritas asteroid family, located in the outer part of the main belt with an average proper semimajor axis of about 3.174 AU. This family has already been studied by applying the spectral formulation of the theorem of Nekhoroshev (1977). The spectral formulation is based on the Fourier analysis of a suitably chosen function of time series of orbital elements obtained by means of the numerical integration covering up to 100 Myr (Guzzo et al., 2002). An example of such a spectrum is shown in Fig. 1 and exhibits a clear continuous structure indicating that the object is not in the Nekhoroshev regime.

The results of the present analysis are given in Table 1. They indicate that Hamilton function for the considered members of Veritas family is NOT quasi-convex, because the eigen values λ_1 and λ_2 of matrix B are of opposite sign. From two last column s of Table 1 one can easily see that values of $\mathcal{K}_1$ corresponding to critical points u_- and u_+ are small (of the order of $\varepsilon^* < 10^{-5}$) for all selected asteroids, except for 2428 Kamenyar, where we get an order of magnitude larger values. Hence, we can conclude that for all asteroids, except for Kamenyar, 3-jet condition is not fulfilled.

Table 1: Calculated Eigen values of B and values of $\mathcal{K}_1$.

Asteroid		Eigen values of B		$\mathcal{K}_1$	
Number	Name	λ_1	λ_2	(u_-)	(u_+)
490	Veritas	-0.13773660D-02	0.67849412D+01	0.2248D-05	0.1518D-05
2147	Kharadze	-0.49082696D-04	0.29421851D-01	0.9742D-05	0.9511D-05
2428	Kamenyar	-0.22328872D-02	0.78520296D+01	0.4225D-04	0.9454D-04
3542	Tanjiazhen	-0.39531112D+00	0.87524627D+01	0.8889D-06	0.5221D-05
52244	1981EE4	-0.24523605D-02	0.69479869D+01	0.4551D-05	0.4948D-05
	2123 PL	-0.41195847D-02	0.70618299D+01	0.4354D-05	0.3382D-05
	4573 PL	-0.41194558D-01	0.68848025D+01	0.8221D-05	0.9377D-05

Thus we can say that we have, for a sample of selected Veritas family members, confirmed results by Guzzo et al. (2002) in a completely independent way.

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