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SPECIFIC DYNAMICAL FEATURES OF MANEV-TYPE PROBLEMS

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Abstract. The Manev-type problems (associated to a potential of the form $A/r + B/r^2$) model many concrete situations belonging to physics and astronomy. The equations of motion and the first integrals of energy and angular momentum are established. Resorting to McGehee's transformations, new, regularized equations of motion are found. Exploiting the rotational symmetry, the phase space dimension is reduced to 3, and clear pictures of the global flow can be obtained. All possible phase curves are surveyed and interpreted in terms of physical orbits. Specific dynamical features (as: motion on precessional conic sections, black hole effect, bounded orbits for nonnegative energy, unstable circular motion, radial librations, etc.) which cannot be met within the framework of classical models are pointed out.

1. INTRODUCTION

One of the most important problems of celestial mechanics is to find a model able to maintain the dynamical astronomy within the framework of classical mechanics (keeping the simplicity and the advantages of the Newtonian model), offering at the same time equally good justifications of the observed phenomena as the relativity. Such a model is that based on the $A/r + B/r^2$ potential (r = distance between particles, A, B = real constants). Newton himself was the first to consider this law, then Clairaut, but the one who based this model on physical principles was the Bulgarian physicist G. Manev (Maneff 1924, 1925, 1930a,b).

Fallen into oblivion for half a century, then pointed out by Hagihara (1975) as providing the same good theoretical approximations as the relativity (at the solar system level, at least), Manev's law and the Manev-type models were recently reconsidered in a series of studies initiated by Diacu (1993). Leaving aside the crucial result (for dynamical astronomy) established within this framework by Lacomba *et al.* (1991), the interest aroused by this problem is proved by the multitude of studies dedicated to the subject or related to. Among them we quote: Casasayas *et al.* (1993), Diacu *et al.* (1995, 1996), Mioc & Stoica (1995a,b,c,d, 1996, 1997a,b), Stoica (1995), Stoica & Mioc (1995, 1996a,b,c), Aparicio and Floria (1996), Delgado *et al.* (1996), Diacu (1996).

The importance of such studies is emphasized by the great variety of concrete physical and astronomical situations modellable via a potential of this kind. Besides classical models (Newton's law, radiative force, force-free field), models as: Fock's field (truncating the negligible terms), Reissner-Nordström field, a photogravitational field (perturbed or not), the two-body problem with equivalent gravitational parameter, the nongravitational homogeneous potentials,the motion of anoutward electron in the field of the nucleus (in a second approximation), etc. also join the Manev-type problems (see Moser 1975; McGehee 1981; Diacu 1990; Şelaru *et al.* 1992, 1993; Mioc & Stoica 1997b).

This paper constitutes itself in a survey of the dynamical properties of the Manev-type two-body problem. On the basis of the powerful tool of McGehee's transformations, the global flow of this problem in a reduced phase space can be fully described. The phase trajectories are translated in terms of physical orbits, then the dynamical features which cannot be met within the framework of classical models are pointed out.

2. EQUATIONS OF MOTION AND FIRST INTEGRALS

Consider the Manev-type two-body problem. We may reduce it to a central force problem (e.g. Diacu *et al.* 1995, 1996; Mioc & Stoica 1997b) and study the motion of one body (hereafter particle) with respect to a fixed frame originated in the other body (hereafter centre). This relative motion will be planar and described by the equation

$$\ddot{\mathbf{r}} = -(A/r^3 + 2B/r^4)\mathbf{r}, \tag{1}$$

where $\mathbf{r}$ = radius vector of the particle with respect to the centre, $r = |\mathbf{r}|$, and dots mark time-differentiation.

In polar coordinates (r, θ), eq.(1) turns to

$$\ddot{r} - r\dot{\theta}^2 = -A/r^2 - 2B/r^3, \tag{2}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0. \tag{3}$$

The angular momentum is conserved, and (3) provides the first integral

$$r^2\dot{\theta} = C, \tag{4}$$

with C = constant of angular momentum. The integral of energy reads

$$\dot{r}^2 + r^2\dot{\theta}^2 = 2A/r + 2B/r^2 + h, \tag{5}$$

with h = constant of energy.

Our qualitative analysis uses the McGehee type transformations (McGehee 1974; for the respective technique applied to less general cases of our problem, see Diacu *et al.* 1995; Delgado *et al.* 1996). We formally multiply (5) by r^2 (a detailed justification of this step was given by Diacu *et al.* 1995), then introduce the transformations

$x = r\dot{r}$ and $y = r^2\dot{\theta}$, and finally rescale the time variable via $dt = r^2 ds$. After simple computations, the system (2)-(3) acquires the form (in which $' = d/ds$):

$$r' = rx, \tag{6}$$

$$\theta' = y,$$

$$x' = r(A + hr),$$

$$y' = 0,$$

with the first integrals of energy and angular momentum given respectively by

$$x^2 + y^2 - 2Ar - hr^2 = 2B; \tag{7}$$

$$y = C. \tag{8}$$

The usefulness of McGehee's transformations is clear: the collision singularity at $r = 0$ was blown up, and new, regularized equations of motion were provided.

3. REDUCED PHASE SPACE

Observe that θ does not appear explicitly in either regularized equations of motion (6) or energy integral (6). We can therefore reduce the 4- dimensional full phase space to dimension 3 (obtaining the reduced phase space RPS) by factorizing the flow to S^1 (recall that $\theta \in S^1$). Exploiting this rotational symmetry, characteristic to our problem, we are able to obtain clear pictures of the global flow.

To describe the flow in RPS, the energy is regarded as a parameter. By (7), one observes that in RPS every energy level (given by a fixed h) is homeomorphic with a quadric surface (nondegenerate or degenerate). The first integral (8) foliates the respective energy level into curves (even degenerate in some cases) lying in the parallel planes $y = C$.

The vector field (6) in RPS (with the equation for θ discarded) exhibits two kinds of equilibria. First, observe that

$$r = 0, \quad x^2 + y^2 = 2B \tag{9}$$

is a circle of equilibria. In other words, under McGehee's transformations, an orbit needs an infinite amount of fictitious time s to reach the collision.

On the other hand, it is easy to see that equilibria outside collision, located at

$$r = -A/h, \quad x = 0, \quad y = \pm\sqrt{2B - A^2/h}, \tag{10}$$

do exist if $2B \geq A^2/h$ and $A/h < 0$.

Diacu *et al.* (1996) depicted the global flow in RPS for the whole allowed interplay among the field parameters (A, B), energy constant (h), and angular momentul (C). Here we shall survey these results with special emphasis on their physical interpretation. Also we shall point out the dynamical features which cannot be met within the framework of the Newtonian model (or of some other classical models).

To characterize the various kinds of phase trajectories in RPS which are met in the Manev-type problem, we shall use the following symbolic notation:

$0 \to 0$: orbits ejecting from collision and then tending back to collision (the particle cannot escape);

$0 \to \infty$: orbits ejecting from collision and tending to infinity;

$0 \to UE$: orbits ejecting from collision and tending to an unstable equilibrium (at distance r_{UE} from the centre);

$UE \to \infty$: orbits ejecting from an unstable equilibrium and tending to infinity;

$UE \to 0$: orbits ejecting from an unstable equilibrium and tending to collision;

$\infty \to UE$: orbits coming from infinity and tending to an unstable equilibrium;

$\infty \to 0$: orbits coming from infinity and tending to collision;

$\infty \to \infty$: orbits coming from infinity and then tending back to infinity (the particle cannot collide with the centre);

UE: unstable equilibrium (at distance r_{UE} from the centre);

SE: stable equilibrium (at distance r_{SE} from the centre);

P: periodic orbits;

$\emptyset$: impossible real motion.

4. PHYSICAL MOTIONS

Before starting our survey, let us specify some general characteristics of the physical trajectories which correspond to the various types of RPS orbits. At the equilibria (10), if $y \neq 0$ ($C \neq 0$) the particle moves in a circular orbit around the centre; if $y = 0$ ($C = 0$) the particle is at rest with respect to the centre.

Outside RPS equilibria, if $y = 0$ the particle moves radially. If $y \neq 0$ the motion has a spiral character: precessional ellipses for $h < 0$; precessional parabolas for $h = 0$; precessional hyperbolas for $h > 0$ (Diacu *et al.* 1995; Delgado *et al.* 1996; Stoica & Mioc 1996c).

Another important feature is the existence of the black hole effect (Diacu *et al.* 1995): spiral collisions/ejections, which occur for $y \neq 0$. The particle spirals infinitely many times around the centre immediately before collision/after ejection.

By (7) and (9), it is clear that for $B < 0$ the motion is collisionless (Diacu *et al.* 1996); this result was also established by Saari (1974), but within a different framework and using another method.

Also, by (5), one observes that for $h < 0$ the motion is bounded (the particle cannot escape). The same formula points out the fact that, when the particle escapes, its asymptotic velocity at infinity is zero for $h = 0$, and positive for $h > 0$.

Lastly, the periodic trajectories in RPS represent in physical space elliptic- type noncollisional orbits. Leaving aside the case $A > 0$, $B = 0$ (for which the orbits are fixed ellipses), these trajectories are quasiperiodic - precessional ellipses which never close, filling densely an annulus, except a set of zero Lebesgue measure of periodic orbits -precessional ellipses, rosette-shaped, which close after a finite number of rotations (Delgado *et al.* 1996; Diacu *et al.* 1996).

Now we can survey the possible physical motions for the whole allowed interplay among A, B, h, and C $(= y)$. We shall keep the symbolic notations introduced at the end of Section 3, because their physical interpretation is now clear.

4. 1. CASE $B > 0$

Consider first $A > 0$ (Manev's or Fock's fields). If $h < 0$, we have

$$|y| > \sqrt{2B - A^2/h} \Rightarrow \emptyset, \quad |y| = \sqrt{2B - A^2/h} \Rightarrow SE,$$
$$\sqrt{2B} < |y| < \sqrt{2B - A^2/h} \Rightarrow P, \quad |y| \leq \sqrt{2B} \Rightarrow 0 \to 0. \tag{11}$$

If $h \geq 0$, then

$$|y| \geq \sqrt{2B} \Rightarrow \infty \to \infty, \quad |y| < \sqrt{2B} \Rightarrow 0 \to \infty, \infty \to 0. \tag{12}$$

Consider now $A = 0$ (inverse-cubic attractive force). For $h < 0$, we have

$$|y| \geq \sqrt{2B} \Rightarrow \emptyset, \quad |y| < \sqrt{2B} \Rightarrow 0 \to 0. \tag{13}$$

If $h = 0$, then

$$|y| > \sqrt{2B} \Rightarrow \emptyset, \quad |y| = \sqrt{2B} \Rightarrow SE, \quad |y| < \sqrt{2B} \Rightarrow 0 \to \infty, \infty \to 0, \tag{14}$$

while for $h > 0$ we get

$$|y| > \sqrt{2B} \Rightarrow \infty \to \infty, \quad |y| \leq \sqrt{2B} \Rightarrow 0 \to \infty, \infty \to 0. \tag{15}$$

Finally, consider $A < 0$ (the most rich case). If $h \leq 0$, we have

$$|y| \geq \sqrt{2B} \Rightarrow \emptyset, \quad |y| < \sqrt{2B} \Rightarrow 0 \to 0. \tag{16}$$

If $0 < h < A^2/(2B)$, then

$$|y| \geq \sqrt{2B} \Rightarrow \infty \to \infty, \quad |y| < \sqrt{2B} \Rightarrow (0 \to 0) + (\infty \to \infty). \tag{17}$$

If $h = A^2/(2B)$, then:

$$|y| \geq \sqrt{2B} \Rightarrow \infty \to \infty, \quad 0 < |y| < \sqrt{2B} \Rightarrow (0 \to 0) + (\infty \to \infty),$$
$$y = 0 \Rightarrow 0 \to UE, UE \to 0, UE \to \infty, \infty \to UE, UE. \tag{18}$$

If $h > A^2/(2B)$, we obtain

$$|y| \geq \sqrt{2B} \Rightarrow \infty \to \infty, \quad \sqrt{2B - A^2/h} < |y| < \sqrt{2B} \Rightarrow (0 \to 0) + (\infty \to \infty),$$
$$|y| = \sqrt{2B - A^2/h} \Rightarrow 0 \to UE, UE \to 0, UE \to \infty, \infty \to UE, UE,$$
$$|y| < \sqrt{2B - A^2/h} \Rightarrow 0 \to \infty, \infty \to 0. \tag{19}$$

4. 2. CASE $B = 0$

Consider $A > 0$ (inverse-square attractive force, e.g. Newton's model). For $h < 0$, we have

$$|y| > A/\sqrt{-h} \Rightarrow \emptyset, \quad |y| = A/\sqrt{-h} \Rightarrow SE, \quad 0 < |y| < A/\sqrt{-h} \Rightarrow P, \\ y = 0 \Rightarrow 0 \to 0. \tag{20}$$

If $h \geq 0$, we find

$$|y| > 0 \Rightarrow \infty \to \infty, \quad y = 0 \Rightarrow 0 \to \infty, \infty \to 0. \tag{21}$$

Consider now $A = 0$ (force-free field). For $h \leq 0$, we have

$$h < 0 \Rightarrow \emptyset, \quad h = 0 \Rightarrow UE, \tag{22}$$

while $h > 0$ leads to

$$|y| > 0 \Rightarrow \infty \to \infty, \quad y = 0 \Rightarrow 0 \to \infty, \infty \to 0. \tag{23}$$

Lastly, put $A < 0$ (inverse-square repelling force, e.g. the radiative force). We have

$$h \leq 0 \Rightarrow \emptyset, \quad h > 0 \Rightarrow \infty \to \infty. \tag{24}$$

4. 3. CASE $B < 0$

Put first $A > 0$. If $h < A^2/(2B)$, the real motion is impossible. For $h = A^2/(2B)$, we distinguish

$$|y| > 0 \Rightarrow \emptyset, \quad y = 0 \Rightarrow SE. \tag{25}$$

If $A^2/(2B) < h < 0$, we have

$$|y| > \sqrt{2B - A^2/h} \Rightarrow \emptyset, \quad |y| = \sqrt{2B - A^2/h} \Rightarrow SE, \quad |y| < \sqrt{2B - A^2/h} \Rightarrow P. \tag{26}$$

For nonnegative energy levels, we obtayin

$$h \geq 0 \Rightarrow \infty \to \infty. \tag{27}$$

Finally, consider $A \leq 0$ (fully repulsive force). We have

$$h \leq 0 \Rightarrow \emptyset, \quad h > 0 \Rightarrow \infty \to \infty. \tag{28}$$

5. SPECIFIC DYNAMICAL FEATURES

We have seen that the Manev-type problems include, as particular cases, different classical models (Newton's law, radiative force, force-free field). In what follows, we shall emphasize only those dynamical features which cannot be met within the framework of these classical situations.

A very important characteristic is the motion on precessional conic sections (for $C \neq 0$). We must point out here the fact that almost all motions on noncollisional precessional ellipses are quasiperiodic (except a set of measure zero of periodic orbits).

Another essential feature is the occurence of the black hole effect. The collisions are much more probable than in the Newtonian model, because the set of initial data leading to them has positive measure.

The existence of bounded orbits for $h \geq 0$ is unusual, too. There are stable circles for $h = 0$ (see (14)), or precessional parabolas and hyperbolas of the type $0 \to 0$ (see (16) and (17)-(19)).

We also have to mention the coexistence, for the same h and for the same C, of the trajectories of the type $0 \to 0$ and $\infty \to \infty$ (see (17)-(19)).

The existence of unstable circular motion for for $C \neq 0$ or unstable rest for $C = 0$ (corresponding to saddles in RPS) deserves to be mentioned, too. This generates radial (see (18) or spiral (see (19)) motions of the types $0 \to UE$, $UE \to 0$, $UE \to \infty$, $\infty \to UE$, which cannot be recovered in classical models.

Another unusual motion is that performed on precessional hyperbolas which turn their convexity to the centre (see (28)).

To end the list of these specific features, we have to emphasize the existence of the stable rest pointed out by (25) and the radial librations described by the last formula (26) for $y = 0$.

This survey provides an insight deep enough in the complexity of Manev-type problems.

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