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IN VELOCITY SPACES THE (UNPERTURBED) PLANETS MOVE IN CIRCLES ONLY

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It is well known that hodographs of motions in 1/r fields are circles, the equation of which is:

$$(v_x - A)^2 + (v_y - B)^2 = C^2$$

where v_x, v_y are the Cartesian components of velocity and A, B, C are the constants of motion determined by initial conditions (Sommerfeld A. 1964). Starting from the conservation laws we here show that the similar and quite useful relation exists between the polar components of velocity, $v_r = dr/dt$ and $v_\varphi = r\,d\varphi/dt$. The laws of conservation of total energy E and angular momentum L for a mass m moving in the field of mass M, $(M >> m)$, read:

$$E = \frac{1}{2}mv_r^2 + \frac{1}{2}mv_\varphi^2 - \gamma\frac{mM}{r} = const \qquad (1)$$

and

$$L = m\,v_\varphi r = const. \qquad (2)$$

If v_φ is expressed from (2) and substituted in (1) one obtains:

$$E = \frac{1}{2}mv_r^2 + \frac{1}{2}mv_\varphi^2 - mwv_\varphi \qquad (3)$$

where we have introduced the new quantity

$$w = \gamma\frac{mM}{L} \qquad (4)$$

which has dimensions of velocity and the meaning of which will become clear later. It is straightforward now to rewrite equation (3) as:

$$v_r^2 + \left(v_\varphi - w\right)^2 = u^2 \qquad (5)$$

where

$$u^2 = \frac{2E}{m} + w^2 = const. \qquad (6)$$

Equation (5) is the equation of a circle of radius u in the v_r, v_φ plane, the center of which is at the point w on v_φ axis (Figure 1.). This is the useful result referred to in the title of this paper.

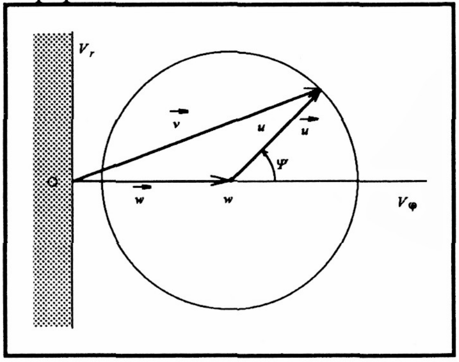

Figure 1. Trajectory in the v_r, v_φ plane. Due to the conservation of angular momentum velocity space reduces to a plane. Angular velocity v_φ can not be negative (no real retrograde motion is possible) and the shaded region is a kinematically forbidden one.

If E and L are once defined by initial conditions, w and u are given by (4) and (6) and the velocity circle is completely specified.

In a point on the circle defined by angle ψ (see Fig.1) the components of the velocity are:

$$\begin{aligned} v_r &= u \sin \psi \\ v_\varphi &= u \cos \psi + w \end{aligned} \qquad (7)$$

Also, if we introduce the vectors $\vec{w}$ and $\vec{u}$ as denoted in Fig .1, it is obvious that

$$\vec{v} = \vec{w} + \vec{u} \quad .$$

The distance r from the center of force at which angular component of velocity equals v_φ is now easily obtained from equations (2) and (7) :

$$r = \frac{L}{mv_\varphi} = \frac{L}{mw(1 + \frac{u}{w}\cos\psi)} \qquad (8)$$

Since u is a fraction of w we may write

$$u = ew \qquad (9)$$

where, with help of (6),

$$e^2 = \frac{2E}{mw^2} + 1 \qquad (10)$$

and equation (8) now becomes:

$$r(\psi) = \frac{p}{1 + e\cos\psi} \qquad (11)$$

This is immediately recognized as the general equation of a conic section in polar coordinates which is usually written as (the polar axis pointing from the pole, at the center of motion, towards the pericenter)

$$r = \frac{p}{1 + e\cos\varphi} \quad . \qquad (12)$$

Direct comparison with equation (8) yields the values of the parameters of the trajectory : the focal parameter $p = L/mw$, the eccentricity $e = u/w$, or as given by equation (10), and, most useful, angle ψ in the velocity diagram is equal to the polar angle φ in the configuration space (or to the so called true anomaly of celestial mechanics). All other elements of the keplerian orbits may now be found from the expressions already deduced, equations (5) to (12).

To demonstrate the ease with which this is done we shall analyze only the basic characteristics of possible motions.

The simplest case corresponds obviously to $u = 0$. Then also $e = 0$, $v_r = 0$ and $v_\varphi = v = w$. Velocity diagram reduces to a point and trajectory is a circle with the radius $r_c = p$ and w is seen to be the velocity on the circular orbit with given L. From equation (1) total energy is seen to be $E = -mw^2/2$ what is its minimum possible value since u can not be negative.

If $1 > e > 0$ the trajectory is an ellipse (equation (12)) and u is smaller then w; the velocity circle does not touch the v_r axis. The characteristic points - the pericenter P, the apocenter A, and the points where the focal parameter touches the trajectory, FP, are marked in Figure 2. All the parameters of the trajectory are easily found and this is left as an exercise to the reader (for instance, velocity at pericenter is instantly seen to be $u + w$, radial velocity is seen to be maximum at FP and to equal u, etc.) Total energy is still negative, as follows from equation (10).

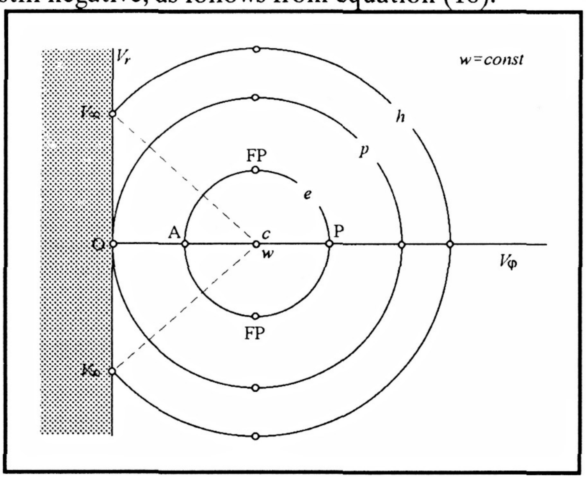

Figure 2. View of different conic section trajectories in velocity plane. Representative circles corresponding to circular (c), elliptic (e), parabolic (p) and hyperbolic (h) trajectories with given angular momentum (same w) are shown. Characteristic points are marked as described in the text.

The discussions of infinite motions equally easy yield the parameters of parabolic and hyperbolic trajectories.

REFERENCES:

Sommerfeld A., 1964, *Mechanics* (New York, Academic Press) p.40 and p.242